TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 211, 1975

ON KŌMURA'S CLOSED-GRAPH THEOREM

BY

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ABSTRACT. Let (α) be a property of separated locally convex spaces. Call a locally convex space $E[\mathfrak{T}]$ an $(\bar{\alpha})$-space if $\mathfrak{T}$ is the final topology defined by $\{u_i: E_i[\mathfrak{T}_i] \to E\}_{i \in I}$, where each $E_i[\mathfrak{T}_i]$ is an (α)-space. Then, for each locally convex space $E[\mathfrak{T}]$, there is a weakest $(\bar{\alpha})$-topology on E stronger that $\mathfrak{T}$, denoted $\mathfrak{T}^{\bar{\alpha}}$.

Kōmura's closed-graph theorem states that the following statements about a locally convex space $E[\mathfrak{T}]$ are equivalent:

(1) For every (α)-space F and every closed linear map $u: F \to E[\mathfrak{T}]$, u is continuous.

(2) For every separated locally convex topology $\mathfrak{T}_0$ on E, weaker than $\mathfrak{T}$, we have $\mathfrak{T} \subset \mathfrak{T}_0^{\bar{\alpha}}$.

Much of this paper is devoted to amplifying Kōmura's theorem in special cases, some well-known, others not.

An entire class of special cases, generalizing Adasch's theory of infra-(s) spaces, is established by considering a certain class of functors, defined on the category of locally convex spaces, each functor yielding various notions of "completeness" in the dual space.

1. Introduction. The principal source of inspiration for this paper is a semester-long series of lectures on closed-graph and open-mapping theorems given by Professor Gottfried Köthe at the University of Maryland in the spring of 1972. During that series of lectures, it occurred to the author that it should be useful if one could show that a locally convex space $E[\mathfrak{T}]$ which is webbed is also strongly webbed, i.e. the space $E[\beta(E, E')]$ is also webbed. It then occurred to him that a transfinite procedure could extend this result to the conclusion that $E[\mathfrak{T}^t]$ is also webbed, where $\mathfrak{T}^t$ is the weakest barreled topology on E stronger than $\mathfrak{T}$. This seemed satisfying, since it was the analogue of a result enunciated by Professor Köthe in his lectures, namely that if $E[\mathfrak{T}]$ is webbed, then so is $E[\mathfrak{T}^\times]$, where $\mathfrak{T}^\times$ is the weakest bornological topology on E stronger than $\mathfrak{T}$. This satisfaction was short-lived, however, since the two results together raise the question of whether they can be simultaneously generalized. The author was able to do this by proving that if $E[\mathfrak{T}]$ is webbed, then so is $E[\mathfrak{T}^u]$, where $\mathfrak{T}^u$ is the weakest

Received by the editors March 7, 1973 and, in revised form, April 10, 1974.
AMS (MOS) subject classifications. Primary 46A30; Secondary 46A07, 46A09.

ultrabornological topology on E stronger than $\mathfrak{T}$. That result is presented in this paper (Theorem 7.1).

In the course of proving the result alluded to above, the author quite naturally never had closed-graph theorems for linear mappings $u: F \to E[\mathfrak{T}]$, F an ultrabornological space, $E[\mathfrak{T}]$ a locally convex space, too far from his mind; and it occurred to him that one could write down a necessary and sufficient condition on $E[\mathfrak{T}]$ for such a theorem to hold, both the condition and the proof being completely nonconstructive, mere symbol-juggling so to speak. The condition is this: For every separated locally convex topology $\mathfrak{T}_0$ on E with $\mathfrak{T}_0 \subset \mathfrak{T}$, we have $\mathfrak{T} \subset \mathfrak{T}_0^u$. Moreover, it occurred to the author that the proof involved worked just as well if one replaced "ultrabornological" with "bornological" or "barreled" or any other property which is stable under the formation of final topologies. When he communicated this insight to Professor Köthe, he was introduced to Kōmura's paper [9], where exactly this general result had appeared over a decade earlier. Kōmura enunciated only one particular case of his general theorem, namely the case where F is barreled. Neither this special case, nor the general case, seems too widely known, though this has changed somewhat recently with the rediscovery of the theorem for F barreled by Adasch [1]. While the new proofs given cannot compare with the simplicity of Kōmura's, they do give the theorem a very substantial advantage it did not have before: The condition on $E[\mathfrak{T}]$ (namely, that $\mathfrak{T} \subset \mathfrak{T}_0^t$ for every separated $\mathfrak{T}_0 \subset \mathfrak{T}$) has now been reformulated in such a way as to be "useful". For instance, one can use the formulations now available to give a quite easy proof of Pták's closed-graph theorem, an advantage sure to be appreciated by those who learned it the hard way. One principal aim of this paper will be to enunciate several special cases of Kōmura's theorem and also to give reformulations of the conditions on $E[\mathfrak{T}]$ that arise in special cases, reformulations which it is to be hoped might impart a feeling of "usefulness" to the result. In the course of doing this, it will be necessary to give at least thumbnail descriptions of the principal consequences of the condition on the space F (the domain). Where the results are well known, the author will so indicate and this will be the case much of the time, though there are some results which are at least not well known to the author to be well known. Also, it is the case that Kōmura's theorem raises some interesting new problems and the author will comment on some of these. At least one of these problems is illuminated in part by the result on webbed spaces mentioned above.

Most notation and terminology will be treated as though it is universally known, the only gesture toward the likely event that this is not the case being occasional short descriptive phrases. For now, we will mention explicitly only the following: The author does not assume that all locally convex topologies are separated. However, the author will use the phrase "locally convex space" only on objects $E[\mathfrak{T}]$, where $\mathfrak{T}$ is a separated locally convex topology on E.

2. **Kōmura's closed-graph theorem.** Let $\{F_i\}_{i\in I}$ be an indexed family of vector spaces and let F be a vector space. For each $i \in I$ suppose we are given a locally convex topology $\mathfrak{S}_i$ on F_i and a linear map $u_i: F_i \to F$. The family of all locally convex topologies $\mathfrak{S}$ on F which make each of the maps $u_i: F_i[\mathfrak{S}_i] \to F[\mathfrak{S}]$ continuous has a least upper bound among all topologies on F which is again locally convex and again makes each of these maps continuous. Hence, there is a strongest locally convex topology $\mathfrak{S}$ on F with this property. $\mathfrak{S}$ is *characterized* by the following property: If E is a vector space, $\mathfrak{T}$ a locally convex topology on E and $u: F \to E$ is a linear map, then u is $\mathfrak{S}-\mathfrak{T}$ continuous if and only if $u \circ u_i$ is $\mathfrak{S}_i - \mathfrak{T}$ continuous for each $i \in I$. $\mathfrak{S}$ is called the *final topology* defined by the family $\{u_i: F_i[\mathfrak{S}_i] \to F\}_{i\in I}$. In case F happens to be the algebraic hull $F = \Sigma_{i\in I} u_i(F_i)$ and in case all the topologies involved, including $\mathfrak{S}$, are separated, $\mathfrak{S}$ is also called the *hull topology*.

Following Kōmura [9], we consider some property (α) pertaining to locally convex topologies and we assume that if $\mathfrak{S}$ is a locally convex topology on a space F then $\mathfrak{S}$ has property (α) only if $\mathfrak{S}$ is separated. If $\mathfrak{S}$ has property (α) we call $\mathfrak{S}$ an (α)-topology and call $F[\mathfrak{S}]$ an (α)-*space*. We can now define a new property $(\bar{\alpha})$ as follows: $F[\mathfrak{S}]$ is an $(\bar{\alpha})$-*space* provided there is some indexed family $\{F_i[\mathfrak{S}_i]\}_{i\in I}$ of (α)-spaces and, for each $i \in I$, a linear map $u_i: F_i \to F$, such that $\mathfrak{S}$ is separated and is the final topology defined by the family $\{u_i: F_i[\mathfrak{S}_i] \to F\}_{i\in I}$. We clearly have $(\alpha) \Rightarrow (\bar{\alpha})$. We say that (α) is *stable under the formation of final topologies* if $(\bar{\alpha}) \Rightarrow (\alpha)$, i.e., $(\alpha) \Leftrightarrow (\bar{\alpha})$. We notice that, whatever (α) is, we have $(\bar{\alpha}) \Leftrightarrow (\bar{\bar{\alpha}})$ by the "transitivity of final topologies". If $\mathfrak{S}$ is the finest locally convex topology on F, then $F[\mathfrak{S}]$ is an $(\bar{\alpha})$-space. (Take the index set I to be empty.)

We now fix a locally convex space $E[\mathfrak{T}]$. We let $\{\mathfrak{T}_i\}_{i\in I}$ be the family of all $(\bar{\alpha})$-topologies on E which are stronger than $\mathfrak{T}$. Then the greatest lower bound of this family among all locally convex topologies on E is stronger than $\mathfrak{T}$, hence is separated. Since it is also the final topology defined by the family of inclusions $\{E[\mathfrak{T}_i] \to E\}_{i\in I}$, it is also an $(\bar{\alpha})$-topology (being separated). Hence, it is the smallest $(\bar{\alpha})$-topology stronger than $\mathfrak{T}$. We denote this topology by $\mathfrak{T}^{\bar{\alpha}}$ and call it the $(\bar{\alpha})$-*topology associated to* $\mathfrak{T}$.

We now state Kōmura's closed-graph theorem:

2.1 **Theorem (Kōmura).** *Let* $E[\mathfrak{T}]$ *be a locally convex space. The following are equivalent*:

(1) *For every* (α)*-space* $F[\mathfrak{S}]$ *and every closed linear map* u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, u *is continuous.*

(2) *For every* $(\bar{\alpha})$*-space* $F[\mathfrak{S}]$ *and every closed linear map* u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, u *is continuous.*

(3) *For every separated locally convex topology* $\mathfrak{T}_0$ *on* E *with* $\mathfrak{T}_0 \subset \mathfrak{T}$, *we have* $\mathfrak{T}_0^{\bar{\alpha}} = \mathfrak{T}^{\bar{\alpha}}$.

(4) *For every separated locally convex topology* $\mathfrak{T}_0$ *on* E *with* $\mathfrak{T}_0 \subset \mathfrak{T}$, *we have* $\mathfrak{T} \subset \mathfrak{T}_0^{\bar{\alpha}}$.

Proof. The proof rests on the following well-known lemma: If $F[\mathfrak{S}]$ and $E[\mathfrak{T}]$ are locally convex spaces and if u: $F \to E$ is linear, then the graph of u is closed in $F[\mathfrak{S}] \times E[\mathfrak{T}]$ (i.e., u is closed) if and only if there is some separated locally convex topology $\mathfrak{T}_0$ with $\mathfrak{T}_0 \subset \mathfrak{T}$ and with u continuous with respect to $\mathfrak{S}$ and $\mathfrak{T}_0$.

(2) $\Rightarrow$ (4): id_E: $E[\mathfrak{T}_0] \to E[\mathfrak{T}]$ is closed. Hence id_E: $E[\mathfrak{T}_0^{\bar{\alpha}}] \to E[\mathfrak{T}]$ is closed. Hence, by (2), id_E: $E[\mathfrak{T}_0^{\bar{\alpha}}] \to E[\mathfrak{T}]$ is continuous. That is, $\mathfrak{T} \subset \mathfrak{T}_0^{\bar{\alpha}}$.

(4) $\Rightarrow$ (2): There is a separated locally convex topology $\mathfrak{T}_0$ with $\mathfrak{T}_0 \subset \mathfrak{T}$ and with u: $F[\mathfrak{S}] \to E[\mathfrak{T}_0]$ continuous. Let $\mathfrak{T}_1$ be the final topology on E defined by u: $F[\mathfrak{S}] \to E$. Then $\mathfrak{T}_0 \subset \mathfrak{T}_1$. Since $\mathfrak{T}_1$ is an $(\bar{\alpha})$-topology, we have $\mathfrak{T} \subset \mathfrak{T}_0^{\bar{\alpha}} \subset \mathfrak{T}_1$. Hence u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$ is continuous.

(1) $\Leftrightarrow$ (2) by the characterization of final topologies.

(3) $\Leftrightarrow$ (4) is trivial. □

2.2 **Corollary (Kōmura).** *Let* $E[\mathfrak{T}]$ *be an* $(\bar{\alpha})$*-space. The following are equivalent*:

(1) *For every* (α)*-space* $F[\mathfrak{S}]$ *and every closed linear map* u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, u *is continuous.*

(2) *For every* $(\bar{\alpha})$*-space* $F[\mathfrak{S}]$ *and every closed linear map* u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, u *is continuous.*

(3) $\mathfrak{T}$ *is minimal among the* $(\bar{\alpha})$*-topologies on* E.

The above corollary raises a problem: If $E[\mathfrak{T}]$ satisfies the statements of the theorem, then does $E[\mathfrak{T}^{\bar{\alpha}}]$ satisfy the statements of the corollary? That is, if $\mathfrak{T}_0$ separated and $\mathfrak{T}_0 \subset \mathfrak{T}$ imply that $\mathfrak{T}_0^{\bar{\alpha}} = \mathfrak{T}^{\bar{\alpha}}$, does it then follow that $\mathfrak{T}^{\bar{\alpha}}$ is minimal among the $(\bar{\alpha})$-topologies on E? This is not at all clear to the author in the general case, nor in many particular cases for that matter. It is the case if $(\alpha) = (\mathcal{B})$: "$F[\mathcal{S}]$ is complete and normable" and $E[\mathcal{T}]$ is webbed. This will be proved later. (See Corollary 7.2.)

Kōmura's theorem itself raises the question as to what stability properties the class of all locally convex spaces $E[\mathcal{T}]$ which satisfy the statements of the theorem has. Let us denote this class by $(\alpha\mathcal{C})$. Then the following facts are clear: If $\mathcal{T}$ is an $(\alpha\mathcal{C})$-topology on E and if $\mathcal{T}_0$ is a separated locally convex topology on E with $\mathcal{T}_0 \subset \mathcal{T}$, then $\mathcal{T}_0$ is an $(\alpha\mathcal{C})$-topology. If $E[\mathcal{T}]$ is an $(\alpha\mathcal{C})$-space and E_0 is a closed subspace of E, then $E_0[\mathcal{T}]$ is an $(\alpha\mathcal{C})$-space. (We confuse $\mathcal{T}$ with $\mathcal{T}|_{E_0}$.) If every (α)-space $F[\mathcal{S}]$ is a Mackey space, i.e., if $\mathcal{S} = \tau(F, F')$, uniform convergence on all absolutely convex, weakly compact subsets of F', and if $E[\mathcal{T}]$ is an $(\alpha\mathcal{C})$-space, then so is $E[\tau(E, E')]$. This suggests an obvious new question: What if every (α)-space $F[\mathcal{S}]$ is barreled, i.e., $\mathcal{S} = \beta(F, F')$, uniform convergence on all weakly bounded subsets of F', and $E[\mathcal{T}]$ is an $(\alpha\mathcal{C})$-space? Does it then follow that $E[\beta(E, E')]$ is an $(\alpha\mathcal{C})$-space? It is instructive to review where the triviality stops here: If u: $F[\mathcal{S}] \to E[\mathcal{T}]$ is continuous, then u is also continuous for $\tau(F, F')$ and $\tau(E, E')$ and for $\beta(F, F')$ and $\beta(E, E')$. So, if $\mathcal{S}$ is a Mackey topology, resp. a barreled topology, the obvious way to try to force a closed-graph theorem for u: $F[\mathcal{S}] \to E[\tau(E, E')]$, resp. for u: $F[\mathcal{S}] \to E[\beta(E, E')]$, is to try to force u to be $\mathcal{S} - \mathcal{T}$ continuous. If $\mathcal{T}$ is an $(\alpha\mathcal{C})$-space, it then suffices to show u is $\mathcal{T}$-closed. But a graph (or a convex set for that matter) is closed in $F[\mathcal{S}] \times E[\mathcal{T}]$ if and only if it is closed in $F[\mathcal{S}] \times E[\tau(E, E')]$. From this, it follows that, if $\mathcal{S}$ is a Mackey space, we have one result. But if $\mathcal{S}$ is barreled, our reasoning breaks down at the following point: A graph may very well be closed in $F[\mathcal{S}] \times E[\beta(E, E')]$, but not closed in $F[\mathcal{S}] \times E[\mathcal{T}]$. The result mentioned above for $(\alpha) = (\mathcal{B})$: complete and normable, and $E[\mathcal{T}]$ webbed shows the situation is not hopeless in special cases, but, if anything, that result is more indicative of the superiority of DeWilde's theory of webbed spaces than an advance in the theory of $(\mathcal{B}\mathcal{C})$-spaces. DeWilde's theory also has other strengths not present to date even in $(\mathcal{B}\mathcal{C})$-spaces: The author has tried, but not succeeded, to prove that a product of two $(\mathcal{B}\mathcal{C})$-spaces is again a $(\mathcal{B}\mathcal{C})$-space. In DeWilde's theory, webbed spaces are stable under the formation of countable hulls and countable projective limits. It would be nice if at least $(\mathcal{B}\mathcal{C})$-spaces could get as far as stability under formation of finite products.

Kōmura's theorem also leaves us with the problem of reformulating statement (4) of the theorem in such a way as to be more useful in special cases. A good part of the rest of this paper will be devoted to doing that. To be specific, the remainder of this paper is organized as follows: In the next four sections we will consider in succession four special cases for the property (α): (α) "universal", true of all locally convex spaces; $(\alpha) = (t)$: "tonnelé" or "barreled"; $(\alpha) = (\mathfrak{N})$: "normable"; $(\alpha) = (\mathfrak{B})$: "complete and normable". In the first two cases, the author has made no contribution of his own, except an expository one. In the latter two cases, it would probably be fair to say that the author has made some original contribution, though it would probably be difficult to pinpoint just where it begins and exposition leaves off.

§7 is devoted to some results due to the author in the theory of webbed spaces, extending some results of DeWilde.

§8 is devoted to a theory of the author's giving a whole class of special cases of Kōmura's theorem, all of which are analogues of Adasch's theorem. §§9 though 12 are devoted to particular applications of this general theory.

3. **On weak, complete spaces.** If the property (α) is "universal", i.e., true of all locally convex spaces, then it is clear that statement (4) of Kōmura's theorem can be reformulated as follows: "$\mathfrak{T}$ is minimal among the separated locally convex topologies on E." The theory of such spaces $E[\mathfrak{T}]$ is well known and easy to develop. We enunciate it by restating Kōmura's theorem in this case:

3.1. **Corollary.** *Let $E[\mathfrak{T}]$ be a locally convex space. The following are equivalent*:

(1) *For every locally convex space $F[\mathfrak{S}]$ and every closed linear map $u: F[\mathfrak{S}] \to E[\mathfrak{T}]$, u is continuous.*

(2) *$\mathfrak{T}$ is minimal among the separated locally convex topologies on E.*

(3) *$E[\mathfrak{T}]$ is a weak, complete space.*

(4) *$E[\sigma(E, E')]$ is complete.*

(5) *$E \approx E'^*$.*

(6) *$\tau(E', E)$ is the strongest locally convex topology on E'.*

(7) *$E[\mathfrak{T}]$ is isomorphic to a product of scalar fields.*

If the above statements hold, then $\sigma(E, E') = \mathfrak{T} = \beta(E, E')$.

4. **On infra-(s) spaces.** We now consider the case where $(\alpha) = (t)$: "barreled". The property (t) is stable under the formation of final topologies, i.e., $(t) \iff (\overline{t})$. The characterization of barreled spaces in which we are interested here is the following: $F[\mathfrak{S}]$ is barreled if and only if $\mathfrak{S}$ is a Mackey topology and $F'[\sigma(F', F)]$ is quasi-complete, i.e., every bounded subset of $F'[\sigma(F', F)]$ is relatively complete in $F'[\sigma(F', F)]$.

For a general locally convex space $E[\mathcal{T}]$, and for any subspace H of E', there is a smallest subspace, denoted $\overline{H}$, of E^*, containing H, such that $\overline{H}[\sigma(\overline{H}, E)]$ is quasi-complete. If H is dense in E', then the topology $\tau(E, \overline{H})$ on E is barreled and is in fact the weakest barreled topology on E which is stronger than $\sigma(E, H)$, itself a separated topology. In particular, we have $\sigma(E, E')^t = \mathcal{T}^t = \tau(E, \overline{E'})$. Otherwise put, $E[\mathcal{T}^t]' = \overline{E'}$.

We now consider statement (4) of Kōmura's theorem (2.1) in this case: Since $\mathcal{T}^t = \sigma(E, E')^t$, we may as well assume all locally convex topologies in question on E are weak. But then, we have $\mathcal{T}_0 \subset \mathcal{T}$ for $\mathcal{T}_0$ separated and weak if and only if $\mathcal{T}_0 = \sigma(E, H)$, where H is a $\sigma(E', E)$ dense subspace of E'. We then have $\mathcal{T} \subset \mathcal{T}_0^t$ if and only if $E' \subset E[\mathcal{T}_0^t]' = \overline{H}$. We have then another (well-known) reformulation of Kōmura's theorem, due to Adasch [1].

4.1 **Corollary.** *Let $E[\mathcal{T}]$ be a locally convex space. The following are equivalent*:

(1) *For every barreled space $F[\mathcal{S}]$ and every closed linear map u: $F[\mathcal{S}] \to E[\mathcal{T}]$, u is continuous.*

(2) *For every separated locally convex topology $\mathcal{T}_0$ on E with $\mathcal{T}_0 \subset \mathcal{T}$, we have $\mathcal{T} \subset \mathcal{T}_0^t$.*

(3) *For every $\sigma(E', E)$-dense subspace H of E', we have $E' \subset \overline{H}$.*

The current name for spaces $E[\mathcal{T}]$ satisfying the statements of this theorem, i.e., for $(t\mathcal{C})$-spaces, is "*infra-(s) spaces*".

5. **On $(\mathfrak{N}\mathcal{C})$-spaces.** We now consider the case where $(\alpha) = (\mathfrak{N})$: "normable". It is definitely not true that $(\mathfrak{N}) \Leftrightarrow (\overline{\mathfrak{N}})$, so our first matter of business is to give a satisfactory reformulation of $(\overline{\mathfrak{N}})$. To this end, we introduce the following notion: If $F[\mathcal{S}]$ is a locally convex space, a set $U \subset F$ will be called a *bornivore*, or *bornivorous*, if U is absolutely convex and absorbs all bounded subsets of $F[\mathcal{S}]$. Clearly every absolutely convex neighborhood of 0 in F is a bornivore. If, conversely, every bornivore is a neighborhood of 0 in F, then we call $F[\mathcal{S}]$ *bornological.* We abbreviate this property by $(\times)$. We have $(\overline{\mathfrak{N}}) \Leftrightarrow (\times)$.

Let $F[\mathcal{S}]$ be a locally convex space and let $B \subset F$ be bounded and absolutely convex. We consider the subspace $\bigcup_{n=1}^{\infty} nB$ of F spanned by B. The Minkowski functional of B, $\|x\|_B = \inf\{1/t : t > 0,\ tx \in B\}$, is a seminorm on this space and, since B can contain no line, it is in fact a norm. We denote by F_B this space, together with this norm. It is a simple matter to check that the final topology defined by the family of insertions $\{F_B \to F\}$ (B bounded, absolutely convex) is the topology $\mathcal{S}^{\times}$.

A sequence $\{y_n\}$ which is contained in some absolutely convex set B for which $\|y_n\|_B \to 0$, is called *a local null sequence* of $F[\mathfrak{S}]$. An absolutely convex set $U \subset F$ is an $\mathfrak{S}^\times$-neighborhood of 0 in F if and only if U absorbs every local null sequence in $F[\mathfrak{S}]$. If, moreover, $E[\mathfrak{T}]$ is another locally convex space, and if $u: F \to E$ is a linear mapping, then u is $\mathfrak{S}^\times - \mathfrak{T}$ continuous if and only if for every local null sequence $\{y_n\}$ in $F[\mathfrak{S}]$, the sequence $\{u(y_n)\}$ is bounded in $E[\mathfrak{T}]$.

In this context, the facts just outlined make the following reformulation of Kōmura's theorem clear:

5.1 **Corollary.** *Let $E[\mathfrak{T}]$ be a locally convex space. The following are equivalent:*

(1) *For every normable space $F[\mathfrak{S}]$ and every closed linear mapping u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, u is continuous.*

(2) *For every bornological space $F[\mathfrak{S}]$ and every closed linear mapping u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, u is continuous.*

(3) *For every locally convex space $F[\mathfrak{S}]$ and every closed linear mapping u: $F[\mathfrak{S}] \to E[\mathfrak{T}]$, $\{u(x_n)\}$ is a bounded sequence in $E[\mathfrak{T}]$ for every local null sequence $\{x_n\}$ in $F[\mathfrak{S}]$.*

(4) *For every separated locally convex topology $\mathfrak{T}_0$ on E with $\mathfrak{T}_0 \subset \mathfrak{T}$, we have $\mathfrak{T} \subset \mathfrak{T}_0^\times$.*

(5) *For every separated locally convex topology $\mathfrak{T}_0 \subset \mathfrak{T}$ and every $\mathfrak{T}_0$-bounded set $B \subset E$, B is $\mathfrak{T}$-bounded.*

(6) *For every separated locally convex topology $\mathfrak{T}_0$ on E with $\mathfrak{T}_0 \subset \mathfrak{T}$ and every local null sequence $\{x_n\}$ in $E[\mathfrak{T}_0]$, $\{x_n\}$ is $\mathfrak{T}$-bounded.*

5.2 **Corollary.** *Let G be a barreled space. The following are equivalent:*

(1) *$G'[\tau(G', G)]$ is an $\mathfrak{M}\mathfrak{C}$-space.*

(2) *Every dense subspace of G is also barreled.*

Proof. Let $E[\mathfrak{T}] = G'[\tau(G', G)]$. Then $G = E'$. A subspace H of $G = E'$ is $\sigma(E', E)$-dense if and only if it is dense in the given topology of G.

Suppose $E[\mathfrak{T}]$ is an $\mathfrak{M}\mathfrak{C}$-space. Let $H \subset G$ be dense. Let U be a barrel in H. Then U^0 is $\sigma(E, H)$-bounded in E, hence $\sigma(E, G)$-bounded. Hence $(U^0)^0$ (formed in G) is a barrel in G, hence a neighborhood of 0 in G. But $U = U^{00} \cap H$, a neighborhood of 0 in H.

Now suppose (2) holds. Let $H \subset G$ be dense and let $B \subset E$ be $\sigma(E, H)$-bounded. Then $B^0 \cap H$ (B^0 formed in G) is a neighborhood of 0 in H. Hence, there is a balanced, convex, $\sigma(E, G)$-compact set $C \subset E$ such that $C^0 \cap H \subset B^0 \cap H$. Hence, $B \subset (B^0 \cap H)^0 \subset (C^0 \cap H)^0 = \overline{C}^{\sigma(E,H)}$. But C is $\sigma(E, G)$-

compact, hence $\sigma(E, H)$-compact, hence $\sigma(E, H)$-closed. So $B \subset C$. So B is bounded. (*Note*: The compactness of C was used at the very end only, in order to proceed from $B \subset \overline{C}^{\sigma(E,H)}$ to $B \subset C$.)

5.3 **Example.** Let $E[\mathcal{T}]$ be $C[0, 1]$ with the uniform topology. This is not an $(\mathcal{NC})$-space. Indeed, there is a sequence $\{x_n\}$ in $C[0, 1]$ with $0 = \lim_{n\to+\infty} x_n(t)$ for all $t \in [0, 1]$ and with $+\infty = \lim_{n\to+\infty} \|x_n\|_\infty$. If B is the absolutely convex hull of this sequence and if $\mathcal{T}_0$ is the topology on $E = C[0, 1]$ of simple convergence on $[0, 1]$, then the insertion $E_B \to E$ is $\mathcal{T}_0$-continuous, hence $\mathcal{T}$-closed. It is not continuous. We have $\mathcal{T} \not\subset \mathcal{T}_0^\times$.

5.4 **Example.** No infinite-dimensional Hilbert space is an $(\mathcal{NC})$-space. Indeed, let E be an infinite-dimensional Hilbert space, let $\{e_i\}_{i\in I}$ be an orthonormal basis in E, and let H be the linear span of $\{e_i\}_{i\in I}$. Then, if $\{i_1, i_2, i_3, \dots\}$ is a sequence in I with distinct terms, it follows that $0 = \lim_{n\to+\infty} ne_{i_n}$ with respect to $\sigma(E, H)$. Yet $\|ne_{i_n}\| = n$ for each $n \geq 1$. Obviously, this example can be generalized to any Banach space with an infinite basis.

5.5 **Example.** We consider the space $L^1(T)$, where T is the unit circle. The space $P(T)$ of all trigonometric polynomials is dense in $L^1(T)'$ with respect to $\sigma(L^1(T)', L^1(T))$. The sequence $\{f_n\}_{n\geq 1}$ in $L^1(T)$, where $f_n(z) = nz^n$, is unbounded in $L^1(T)$, but converges to 0 with respect to $\sigma(L^1(T), P(T))$. So $L^1(T)$ is not an $(\mathcal{NC})$-space.

The next example is due to Eberhardt. (See Eberhardt [6, §5, Beispiel 1].) It came to the author's attention some months after the first draft of this paper was prepared. It gives us a locally convex space which is not weakly complete, but which is an $(\mathcal{NC})$-space. The author's inability for quite a while to think of such an example accounts for many of the examples in this section and the next. Eberhardt's example is rather embarassingly simple:

5.6 **Example.** Let I be an uncountable set and let $E[\mathcal{T}]$ be the subspace of the locally convex product $\mathbf{K}^I$ comprised of all $x \in \mathbf{K}^I$ for which $\{i: x_i \neq 0\}$ is countable. If $\{x^{(n)}\}$ is any sequence with terms in E, then there is a countable set $J \subset I$ such that, if $\mathbf{K}_J = \{x \in \mathbf{K}^I: x_i \neq 0 \Rightarrow i \in J\}$, then $x^{(n)} \in \mathbf{K}_J \subset E$ for all n. But, with its subspace topology, $\mathbf{K}_J \approx \mathbf{K}^J$ and so, if $\mathcal{T}_0 \subset \mathcal{T}$ is a separated locally convex topology on E, the induced topology on $\mathbf{K}_J$ by $\mathcal{T}_0$ is just that induced by $\mathcal{T}$. So, if $\{x^{(n)}\}$ is $\mathcal{T}_0$-bounded, it is $\mathcal{T}$-bounded. So $E[\mathcal{T}]$ is an $(\mathcal{NC})$-space. It is not weakly complete.

6. On $(\mathcal{BC})$-spaces.

We now consider the most classical of our cases, $(\alpha) = (\mathcal{B})$: "complete and normable". It is most certainly not the case that

$(\mathfrak{B}) \Leftrightarrow (\bar{\mathfrak{B}})$. We note that since $(\mathfrak{B}) \Rightarrow (\mathfrak{M})$, we have $(\bar{\mathfrak{B}}) \Rightarrow (\bar{\mathfrak{M}}) \Leftrightarrow (\times)$ and so the $(\bar{\mathfrak{B}})$-spaces form a subclass of the bornological spaces. Hence, the standard name for this class is rather natural: A $(\bar{\mathfrak{B}})$-space is called an *ultrabornological* space. In place of the notation $(\bar{\mathfrak{B}})$ we will use (u).

Let $E[\mathfrak{T}]$ be a locally convex space. A bounded, absolutely convex subset B of E will be called a *Banach disc* of $E[\mathfrak{T}]$ if E_B is a Banach space. Notice that the only role $\mathfrak{T}$ plays in this definition is to insure B bounded, so that the insertion $E_B \to E[\mathfrak{T}]$ is continuous. We remark that if B is sequentially complete for any other separated locally convex topology on E for which B is also bounded, then B is a Banach disc for $E[\mathfrak{T}]$. In particular, if $E[\mathfrak{T}]$ is quasi-complete and B is closed, then B is a Banach disc. If B is $\sigma(E, E')$-compact, then B is a Banach disc. We denote by $\mathfrak{B}_{\mathfrak{T}}$ the family of all Banach discs for $E[\mathfrak{T}]$. We denote by $\mathfrak{K}_{\mathfrak{T}}$ the family of all absolutely convex, $\mathfrak{T}$-compact subsets of E. We now consider the topologies on E which are, respectively, the final topologies defined by the families

$$\{E_B \to E\}_{B \in \mathfrak{B}_{\mathfrak{T}}}, \quad \{E_K \to E\}_{K \in \mathfrak{K}_{\sigma(E,E')}}, \quad \{E_K \to E\}_{K \in \mathfrak{K}_{\mathfrak{T}}}.$$

It is an important fact, and not difficult to prove, that each of these structures is $\mathfrak{T}^u$. We can reformulate this in terms of absolutely convex sets $U \subset E$: We have U a $\mathfrak{T}^u$-neighborhood of $0 \Leftrightarrow U$ absorbs each set in $\mathfrak{B}_{\mathfrak{T}}$ $\Leftrightarrow U$ absorbs each set in $\mathfrak{K}_{\sigma(E,E')} \Leftrightarrow U$ absorbs each set in $\mathfrak{K}_{\mathfrak{T}} \Leftrightarrow U$ absorbs each set in $\mathfrak{K}_{\mathfrak{T}^u}$. The last test mentioned is the least stringent, although it has an aspect of circularity to it, since it uses $\mathfrak{T}^u$ to describe $\mathfrak{T}^u$. It is useful, however, in that in place of $\mathfrak{K}_{\mathfrak{T}^u}$ we can use $\mathfrak{K}_{\mathfrak{T}'}$, for any $\mathfrak{T}' \subset \mathfrak{T}^u$, for instance $\mathfrak{T}' = \beta(E, E')$. A sequence $\{y_n\}$ in $E[\mathfrak{T}]$ is said to be *fast convergent* to $y_0 \in E$ if there is some set $K \in \mathfrak{K}_{\mathfrak{T}}$ such that $y_n \in K$ for all $n \geq 0$ and such that $|y_n - y_0|_K \to 0$. An absolutely convex set $U \subset E$ is a $\mathfrak{T}^u$-neighborhood of 0 if U absorbs every fast convergent null sequence of $E[\mathfrak{T}^u]$ and only if U absorbs every fast convergent null sequence of $E[\sigma(E, E')]$. We can rephrase this in terms of linear maps v: $E[\mathfrak{T}] \to H[\mathfrak{R}]$, where $H[\mathfrak{R}]$ is a locally convex space: v is $\mathfrak{T}^u - \mathfrak{R}$ continuous if and only if $\{v(y_n)\}$ is bounded in $H[\mathfrak{R}]$ for every fast convergent null sequence $\{y_n\}$ of $E[\mathfrak{T}']$, where $\mathfrak{T}'$ is some (any) separated locally convex topology on E with $\sigma(E, E') \subset \mathfrak{T}' \subset \mathfrak{T}^u$.

In this context, the facts just outlined make the following reformulation of Kōmura's theorem clear:

6.1 **Corollary.** *Let* $E[\mathfrak{T}]$ *be a locally convex space. The following are equivalent*:

(1) *For every* ($\mathfrak{B}$)*-space* $F[\mathfrak{S}]$ *and every closed linear mapping* $u: F[\mathfrak{S}] \to E[\mathfrak{T}]$, u *is continuous.*

(2) *For every ultrabornological space* $F[\mathfrak{S}]$ *and every closed linear mapping* $u: F[\mathfrak{S}] \to E[\mathfrak{T}]$, u *is continuous.*

(3) *For every locally convex space* $F[\mathfrak{S}]$ *and every closed linear mapping* $u: F[\mathfrak{S}] \to E[\mathfrak{T}]$, *we have* $\{u(x_n)\}$ *bounded in* $E[\mathfrak{T}]$ *for every fast convergent null sequence* $\{x_n\}$ *in* $F[\mathfrak{S}]$.

(4) *For every separated locally convex topology* $\mathfrak{T}_0$ *on* E *with* $\mathfrak{T}_0 \subset \mathfrak{T}$, *we have* $\mathfrak{T} \subset \mathfrak{T}_0^u$.

(5) *For every separated locally convex topology* $\mathfrak{T}_0$ *on* E *with* $\mathfrak{T}_0 \subset \mathfrak{T}$ *and every* $\mathfrak{T}_0$*-compact, absolutely convex set* $K \subset E$, K *is* $\mathfrak{T}$*-bounded.*

(6) *For every separated locally convex topology* $\mathfrak{T}_0$ *on* E *with* $\mathfrak{T}_0 \subset \mathfrak{T}$ *and every fast convergent null sequence* $\{x_n\}$ *in* $E[\mathfrak{T}_0]$, $\{x_n\}$ *is* $\mathfrak{T}$*-bounded.*

6.2 **Corollary.** *Let* G *be a barreled space. The following are equivalent:*

(1) $G'[\tau(G', G)]$ *is a* ($\mathfrak{BC}$)*-space.*

(2) *Every dense subspace of* G *is a Mackey space.*

Proof. The proof is nearly identical with that of Corollary 5.2, with only two exceptions: Instead of considering a barrel U in H, one must consider a closed, absolutely convex Mackey neighborhood of 0. Instead of considering a $\sigma(E, H)$-bounded set $B \subset E$, one must consider a balanced, convex, $\sigma(E, H)$-compact set.

6.3 **Corollary.** *Let* G *be a Frêchet space. The Mackey dual* $G'[\tau(G', G)]$ *is a* ($\mathfrak{BC}$)*-space.*

Hence, in particular, every reflexive ($\mathfrak{B}$)-space is a ($\mathfrak{BC}$)-space. (The classical theory gives much stronger results in this regard, but this particular proof is still surprising.)

6.4 **Example.** Let X be a compact topological space and let $C(X)$ denote the space of continuous scalar-valued functions on X, endowed with the structure of uniform convergence on X. If a subset K of $C(X)$ is absolutely convex and is compact in the topology of simple convergence on X, then K is uniformly bounded. (Hence K is weakly compact by a theorem of Grothendieck.) The reason for this is that $C(X)$ is a ($\mathfrak{BC}$)-space by the classical closed-graph theorem. To prove the result directly, put $E = C(X)$ and apply the classical theorem to the (closed) insertion $E_K \to C(X)$.

6.5 **Example.** Let $E = C[0, 1]$ and let $\mathfrak{T}$ denote the topology on E of uniform convergence. Let $\mathfrak{T}_0$ denote the topology on E of simple convergence on $[0, 1]$. There is *no* sequentially-complete separated locally convex topology on E which is weaker than $\mathcal{T}_0$. For if $\mathcal{T}_1$ were such a topology and if $B \subset E$ were $\mathcal{T}_0$-bounded, then the $\mathcal{T}_1$-closed, absolutely convex hull B_1 of B in E would be a $\mathcal{T}_1$-Banach disc, hence bounded for $\mathcal{T}_1^u \supset \mathcal{T}$. Hence B would be $\mathcal{T}$-bounded. But we saw in Example 5.3 that there is a $\mathcal{T}_0$-bounded set which is not $\mathcal{T}$-bounded.

Note in particular that there is no topology $\mathcal{T}_1$ on E with $\mathcal{T}_1 \subset \mathcal{T}_0$ and $\mathcal{T}_1$ minimal among all locally convex topologies on E. While this example is a negative one, it does contain the proof, in essence, of the following positive result.

6.6 **Corollary.** *Let $E[\mathcal{T}]$ be a $(\mathcal{BC})$-space and let $F[\mathcal{S}]$ be a bornological space. Let u: $F[\mathcal{S}] \to E[\mathcal{T}]$ be a linear map and suppose there is some sequentially complete separated locally convex topology $\mathcal{T}_0$ on E with $\mathcal{T}_0 \subset \mathcal{T}$ and with u $\mathcal{S} - \mathcal{T}_0$ continuous. (In this case, u is closed.) Then u is continuous.*

Proof. u is $\mathcal{S} - \mathcal{T}_0$ continuous. So it suffices to show $\mathcal{T} \subset \mathcal{T}_0^\times$. But since $E[\mathcal{T}_0]$ is sequentially complete, we have $\mathcal{T}_0^\times = \mathcal{T}_0^u$. Since $E[\mathcal{T}]$ is a $(\mathcal{BC})$-space, $\mathcal{T} \subset \mathcal{T}_0^u$. □

6.7 **Example.** We denote by ϕ the space of all scalar sequences $a = (a_1, a_2, \dots)$ which are 0 except for a finite number of terms. With its stongest locally convex topology, it is webbed. Hence, ϕ is a $(\mathcal{BC})$-space with respect to every separated locally convex topology on ϕ. We choose for our initial topology $\mathcal{T}$ on ϕ the topology $\mathcal{T} = \sigma(\phi, \phi)$ induced by the canonical bilinear form on $\phi \times \phi$: $\langle a, b\rangle = \sum_{i=1}^\infty a_i b_i$. Since $\phi'^* \approx \phi^* = \omega$, the countable product of scalar fields, it is clear that there are noncontinuous linear forms on $\phi' = \phi'[\sigma(\phi', \phi)] \approx \phi$. We choose the obvious one, that induced by the sequence $e = (1, 1, 1, \dots) \in \omega$. We denote it by

$$H = \operatorname{Ker} e = \left\{ b \in \phi' \approx \phi \colon \sum_{i=1}^{\infty} b_i = 0 \right\} \subset \phi'.$$

Then H is $\sigma(\phi', \phi)$-dense in ϕ', hence $\mathcal{T}_0 = \sigma(\phi, H)$ is a separated locally convex topology on ϕ weaker than $\mathcal{T}$. We now consider the sequence $a^{(1)}$, $a^{(2)}, \dots$ in ϕ defined as follows:

$$a_i^{(j)} = \begin{cases} j, & \text{if } i \le j, \\ 0, & \text{if } i > j. \end{cases}$$

Then, if $b \in H$, and if $n \geq 1$ is chosen so that $b_i = 0$ if $i > n$, then we have, for all $m \geq n$,

$$\langle a^{(m)}, b\rangle = \sum_{i=1}^{+\infty} mb_i = m \sum_{i=1}^{+\infty} b_i = 0.$$

Hence, $\lim_{n\to+\infty} a^{(n)} = 0$ w.r.t. $\mathcal{T}_0$. Of course, since $\mathcal{T}$ is simply the induced product topology on ϕ, a subset of ϕ is $\mathcal{T}$-bounded if and only if it is bounded in each coordinate. Hence $\{a^{(n)}\}_n$ is not $\mathcal{T}$-bounded. Moreover, since the product topology on ω makes ω an (F)-space, $\mathcal{T}$ is metrizable, hence bornological. So we have $\mathcal{T}_0 \subset \mathcal{T}_0^{\times} \subsetneq \mathcal{T} = \mathcal{T}^{\times}$. Notice that $\mathcal{T}_0^{\times}$ is a bornological topology on ϕ, weaker than $\mathcal{T}$, and that *no* coordinate projection is continuous with respect to $\mathcal{T}_0^{\times}$, since $\{a^{(n)}\}_n$ is $\mathcal{T}_0^{\times}$-bounded, but bounded in *no* coordinate.

Of course, the above paragraph shows that $\phi[\mathcal{T}]$ is not an $(\mathcal{NC})$-space. It is also possible to argue this in a less constructive, but more emphatic, fashion: Let F be *any* subspace of ϕ' and let K be *any* $\sigma(F, \phi)$-compact, absolutely convex subset of F. Then K is also $\sigma(\phi', \phi)$-compact and absolutely convex in ϕ'. However, $\phi'[\sigma(\phi', \phi'^*)] \approx \phi[\sigma(\phi, \omega)]$ is a $(\mathcal{BC})$-space and $\sigma(\phi', \phi) \subset \sigma(\phi', \phi'^*)$ on ϕ'. Hence K is also $\sigma(\phi', \phi'^*)$-bounded. But a bounded subset of $\phi'[\sigma(\phi', \phi'^*)] \approx \phi[\sigma(\phi, \omega)]$ must be contained in $\phi'_n = \{b \in \phi' \colon b_i = 0 \text{ for } i > n\}$ for some $n \geq 1$. Hence $K \subset F_n = F \cap \phi'_n$ for some $n \geq 1$. But F_n, being finite-dimensional, admits a countable fundamental system $\mathcal{K}_n$ of compact sets. Hence $\mathcal{K} = \bigcup_{n=1}^{+\infty} \mathcal{K}_n$ is a countable family of $\sigma(F, \phi)$-compact, finite-dimensional subsets of F and K is contained in some set of $\mathcal{K}$, this statement holding for *any* $\sigma(F, \phi)$-compact, absolutely convex subset K of F, for *any* subspace F of ϕ'. This proves the following statement: If $\mathcal{T}_1$ is *any* separated locally convex topology on ϕ with $\mathcal{T}_1 \subset \mathcal{T}$, then $\mathcal{T}_1$ is a weak topology and $\mathcal{T}_1$ is metrizable, hence $\mathcal{T}_1 = \mathcal{T}_1^{\times}$. Therefore, $\phi[\mathcal{T}_1]$ is an $(\mathcal{NC})$-space if and only if $\mathcal{T}_1$ is minimal among all separated locally convex topologies on ϕ, hence if and only if $\phi[\mathcal{T}_1]$ is an (F)-space. But the only topology on ϕ for which ϕ is ultrabornological is the strongest locally convex topology on ϕ. Hence, $\phi[\mathcal{T}_1]$ is not an $(\mathcal{NC})$-space for *any* separated locally convex topology $\mathcal{T}_1$ on ϕ which is separated and weaker than $\mathcal{T}$! □

We give one last "positive" result in this section:

6.8 **Corollary.** *Let $E[\mathcal{T}]$ be a locally convex space. We assume that the following three conditions hold*: (1) *$E[\mathcal{T}]$ is a $(\mathcal{BC})$-space.* (2) *There is some topology $\mathcal{T}_0 \subset \mathcal{T}$ which is minimal among the separated locally convex topologies on E.* (3) *The algebraic dimension of $E[\mathcal{T}_0]'$ is less than the first strongly inaccessible cardinal. Our conclusion is that $\mathcal{T} = \mathcal{T}_0$.*

Proof. $E[\mathfrak{T}_0]$ is isomorphic to a product of scalar fields, the number of factors being equal to the algebraic dimension of $E[\mathfrak{T}_0]'$. Hence $E[\mathfrak{T}_0]$ is ultrabornological. But $E[\mathfrak{T}_0] \to E[\mathfrak{T}]$ is closed, hence continuous. So $\mathfrak{T}_0 \supset \mathfrak{T}$. (*Note*: It was immaterial here whether the cardinality condition was put on $\dim E[\mathfrak{T}_0]'$, $\dim E[\mathfrak{T}]'$ or $\dim E[\mathfrak{T}]$.) □

7. **On webbed spaces.** Let E be a vector space. Let N be a countable index set. Suppose that, for every finite sequence $(n_1, \ldots, n_k)$ with terms in N, we are given a subset $C_{n_1 \ldots n_k} \subset E$ in such a way that we have $E = \bigcup_{n_1} C_{n_1}$ and $C_{n_1 \ldots n_k} = \bigcup_{n_{k+1}} C_{n_1 \ldots n_k n_{k+1}}$. Then the indexed family $\mathfrak{W} = \{C_{n_1 \ldots n_k}\}_{n_1 \ldots n_k}$ is called a *web* on E. If $\mathfrak{T}$ is a separated locally convex topology on E, we call $\mathfrak{W}$ a $\mathcal{C}$*-web* for $\mathfrak{T}$ provided that the following condition holds: For every sequence $(n_1, n_2, n_3, \ldots)$ with terms in N there is a sequence $(\rho_1, \rho_2, \rho_3, \ldots)$ of strictly positive scalars such that for every sequence $(x_1, x_2, x_3, \ldots)$ with terms in E and every scalar sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ such that $x_k \in C_{n_1 \ldots n_k}$ and $0 \leq \lambda_k \leq \rho_k$ for $k \geq 1$, the series $\Sigma_{k=1}^{+\infty} \lambda_k x_k$ is conditionally summable in $E[\mathfrak{T}]$. We notice that, if this is the case, then we can also conclude that $\Sigma_{k=1}^{+\infty} \lambda_k x_k$ is conditionally convergent under the weaker hypothesis that $|\lambda_k| \leq \rho_k$ for $k \geq 1$. Indeed, if $(\lambda_1, \lambda_2, \ldots)$ is a sequence of real scalars with this property, we need only consider $(\lambda_1^+, \lambda_2^+, \ldots)$ and $(\lambda_1^-, \lambda_2^-, \ldots)$. If $(\lambda_1, \lambda_2, \ldots)$ is a sequence of complex scalars, we need then only consider $(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2, \ldots)$ and $(\mathrm{Im}\,\lambda_1, \mathrm{Im}\,\lambda_2, \ldots)$. If $E[\mathfrak{T}]$ is an arbitrary locally convex space, we call $E[\mathfrak{T}]$ a *webbed space* if there is some web on E which is a $\mathcal{C}$-web for $\mathfrak{T}$.

The theory of webbed spaces is due to DeWilde and the principal reference is DeWilde [3], [4]. The principal reason for their invention is that one can prove the following theorems, due to DeWilde:

(1) If $F[\mathfrak{S}]$ is a $(\mathfrak{B})$-space and $E[\mathfrak{T}]$ a webbed space and if $u: F \to E$ is a linear map, the graph of which is sequentially closed in $F[\mathfrak{S}] \times E[\mathfrak{T}]$, then u is continuous.

(2) If $F[\mathfrak{S}]$ is a Baire space and $E[\mathfrak{T}]$ is a webbed space, then every closed linear map $u: F[\mathfrak{S}] \to F[\mathfrak{T}]$ is continuous. In particular, the webbed spaces form a subclass of the $(\mathfrak{B}\mathcal{C})$-spaces. The theory of DeWilde has many very striking strengths. We mention only some of the more outstanding ones: Every (F)-space is webbed. The strong dual of every (F)-space is webbed. Every countable hull of webbed spaces is webbed. Every closed subspace of a webbed space is webbed. Every countable product of webbed spaces is webbed. Perhaps the most striking of all aspects of DeWilde's theory is that the proofs of the two results listed above are essentiallly just Banach's

proof of the closed-graph theorem in the classical case!

The principal result of this section will follow just below. If it is not known in precisely the form given here, then it is at least known in some sense. See, for instance, DeWilde [5, Example 2, p. 58]. (The author wishes to thank the referee for making him aware of this result of DeWilde.) The crucial step in any proof of the following theorem is to show that, given a $\mathcal{C}$-web, the condition on the scalars can be made stringent enough to show that the series involved are not only convergent, but fast-convergent. From this, it will follow that they are in fact $\mathcal{T}^u$-convergent. The argument can then be repeated to show that the condition on the scalars can be made so stringent as to yield series which are in fact fast-convergent with respect to $\mathcal{T}^u$. The technique used by DeWilde to show that the condition on the scalars can be made so stringent as to yield fast-convergent series is a slight modification of a result which can be found in Köthe [8, §20, 9. (6)] and this approach was suggested to the author as an alternative to the proof given here by Professor Köthe. The proof given here is the author's own.

7.1 **Theorem.** *Suppose* $\mathfrak{W}$ *is a* $\mathcal{C}$*-web for* $E[\mathcal{T}]$. *Then, for every sequence* $n = (n_1, n_2, n_3, \dots)$ *of indices, there is a sequence* $\sigma(n) = \sigma = (\sigma_1, \sigma_2, \dots)$ *of strictly positive scalars such that, for each sequence* $x = (x_1, x_2, \dots)$ *with terms* $x_k \in C_{n_1 \dots n_k}$ *and each scalar sequence* $\lambda = (\lambda_1, \lambda_2, \dots)$ *with* $|\lambda_k| \le \sigma_k$ *for all* $k \ge 1$, *the series* $\sum_{k=1}^{+\infty} \lambda_k x_k$ *is fast-convergent in* $E[\mathcal{T}^u]$.

Among the locally convex topologies on E *which are stronger than* $\mathcal{T}$, $\mathcal{T}^u$ *is the weakest which is ultrabornological and the strongest which is webbed.*

Proof. For each sequence $n = (n_1, n_2, \dots)$ of indices, we fix a sequence $\rho = \rho(n) = (\rho_1, \rho_2, \dots)$ of strictly positive numbers such that whenever $\lambda = (\lambda_1, \lambda_2, \dots)$ is a sequence of scalars with $|\lambda| \le \rho$ and whenever $x = (x_1, x_2, \dots)$ is a sequence in E with $x_k \in C_{n_1 \dots n_k}$ for all $k \ge 1$, then the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges conditionally in $E[\mathcal{T}]$. For each sequence $n = (n_1, n_2, \dots)$ of indices, each sequence $x = (x_1, x_2, \dots)$ with terms in E such that $x_k \in C_{n_1 \dots n_k}$ for all $k \ge 1$, and each number r with $0 < r < 1$, we define a function

$$\psi_{nxr}: l\infty \to E$$

$$a \mapsto \sum_{k=1}^{+\infty} a_k r^k \rho_k x_k.$$

Then ψ_{nxr} is linear. If U denotes the closed unit square of $l\infty$, we denote $B_{nxr} = \psi_{nxr}(U)$. Then B_{nxr} is absolutely convex and $E = \bigcup_{n,x,r} B_{nxr}$.

We make the following claims:

(1) B_{nxr} is $\mathfrak{T}$-compact for all n, x, r.

(2) Let $\hat{\mathfrak{T}}$ denote the hull topology on E defined by the family of insertions $\{E_{B_{nxr}} \to E\}_{n,x,r}$. Then $\hat{\mathfrak{T}}$ is ultrabornological and $\mathfrak{T} \subset \hat{\mathfrak{T}}$.

(3) $\mathfrak{W}$ is a $\mathcal{C}$-web for $\hat{\mathfrak{T}}$. To be specific, if $m = (m_1, m_2, \ldots)$ is a sequence of indices, if $\rho = \rho(m)$, if $0 < s < 1$, if $t = s^2$, and if $y_k \in C_{m_1 \ldots m_k}$ and $0 \le \lambda_k \le t^k \rho_k$ for all $k \ge 1$, then we have

$$y_0 - \sum_{k=1}^{k_0} \lambda_k y_k \in s^{k_0+1} B_{mys}$$

for all $k_0 \ge 1$, where y_0 is the sum of the series $\Sigma_{k=1}^{+\infty} \lambda_k y_k$ in $E[\mathfrak{T}]$.

(4) B_{nxr} is $\hat{\mathfrak{T}}$-compact for all n, x, r. Hence, if $m = (m_1, m_2, \ldots)$, if $\rho = \rho(m)$, if $0 < t < 1$, and if $y_k \in C_{m_1 \ldots m_k}$ and $0 \le \lambda_k \le t^k \rho_k$ for all $k \ge 1$, then we have the sequence $\{\Sigma_{i=1}^{k} \lambda_i y_i\}_k$ fast-convergent to $\Sigma_{i=1}^{+\infty} \lambda_i y_i$ with respect to $\hat{\mathfrak{T}}$.

(5) If $\mathfrak{T}'$ is a locally convex topology on E with $\mathfrak{T} \subset \mathfrak{T}'$ and $E[\mathfrak{T}']$ webbed, then $\mathfrak{T}' \subset \hat{\mathfrak{T}}$.

(6) If $\mathfrak{T}'$ is a locally convex topology on E with $\mathfrak{T} \subset \mathfrak{T}'$ and $E[\mathfrak{T}']$ ultrabornological, then $\hat{\mathfrak{T}} \subset \mathfrak{T}'$. That is, $\hat{\mathfrak{T}} = \mathfrak{T}^u$.

We now establish these claims, thus completing the proof:

(1) Let $u \in E[\mathfrak{T}]'$. Then the scalar series

$$\sum_{k=1}^{\infty} a_k u(r^k \rho_k x_k) = u\left(\sum_{k=1}^{\infty} a_k r^k \rho_k x_k\right)$$

converges for all $a \in l\infty$. Hence, $(u(r^k \rho_k x_k))_k$ is in l_1 and is the image of u under the algebraic adjoint ψ^*_{nxr}. Hence ψ_{nxr} is $\sigma(l\infty, l_1)$-$\sigma(E, E[\mathfrak{T}]')$ continuous. Since U is $\sigma(l\infty, l_1)$-compact, it follows that $B_{nxr} = \psi_{nxr}(U)$ is $\sigma(E, E[\mathfrak{T}]')$-compact. But, if $0 < s < 1$ and $s^2 = r$, then, for all $a \in U$, $\Sigma_{k=1}^{+\infty} a_k r^k \rho_k x_k$ is in the closed, absolutely convex hull of the sequence $s(s^k \rho_k x_k)_k/(1 - s)$ in E, since

$$a_k r^k \rho_k x_k = (a_k s^k)(s^k \rho_k x_k)$$

and

$$\sum_{k=1}^{\infty} |a_k| s^k \le \sum_{k=1}^{\infty} s^k = \frac{s}{1-s}.$$

But the sequence $(s^k \rho_k x_k)$ is conditionally summable, hence precompact in $E[\mathfrak{T}]$. (Actually, the sequence $(\rho_k x_k)_k$ is also precompact in $E[\mathfrak{T}]$, but the argument is phrased so as to be valid, word for word, later on with $\hat{\mathfrak{T}}$ in place of $\mathfrak{T}$.) But then, the closed, absolutely convex hull of $s(s^k\rho_k x_k)_k/(1-s)$ in $E[\mathfrak{T}]$ is also $\mathfrak{T}$-precompact. Hence B_{nxr} is $\mathfrak{T}$-precompact. Being weakly complete, it is therefore $\mathfrak{T}$-compact.

(2) trivial.

(3) The proof is practically written into the statement: Choose $a \in U^+$ such that $\lambda_k = a_k t^k \rho_k$ for all $k \geq 1$. Then

$$\begin{aligned} y_0 - \sum_{k=1}^{k_0} \lambda_k y_k &= \sum_{k>k_0} a_k t^k \rho_k y_k \\ &= \sum_{k>k_0} (a_k s^k) s^k \rho_k y_k \\ &= s^{k_0+1} \sum_{k>k_0} (a_k s^{k-k_0-1}) s^k \rho_k y_k \\ &\in s^{k_0+1} \psi_{mys}(U). \end{aligned}$$

(4) The proof of statement (1) was intentionally written in such a way as to be valid for $\hat{\mathfrak{T}}$ in place of $\mathfrak{T}$, given that $\Sigma_{k=1}^{+\infty} a_k r^k \rho_k x_k$ is $\hat{\mathfrak{T}}$-convergent for all $a \in U$ and that, if $s^2 = r$, then $\Sigma_{k=1}^{+\infty} s^k \rho_k x_k$ is also $\hat{\mathfrak{T}}$-convergent. Statement (3) implies both these latter statements. Statement (3) then, together with the compactness statement, implies the statement about fast-convergence.

(5) Apply DeWilde's closed-graph theorem to $E[\hat{\mathfrak{T}}] \to E[\mathfrak{T}']$.

(6) Apply DeWilde's closed-graph theorem to $E[\mathfrak{T}'] \to E[\hat{\mathfrak{T}}]$, using the fact that $E[\hat{\mathfrak{T}}]$ is webbed. □

7.2 **Corollary.** *Suppose* $\mathfrak{W}$ *is a* $\mathcal{C}$*-web for* $E[\mathfrak{T}]$.

(1) *If* $\mathfrak{T}'$ *is any separated locally convex topology on* E *with* $\mathfrak{T}' \subset \mathfrak{T}^u$, *then* $\mathfrak{W}$ *is a* $\mathcal{C}$*-web of* $E[\mathfrak{T}']$. *In particular, this holds for* $\mathfrak{T}' = \beta(E, E')$, *for* $\mathfrak{T}' = \mathfrak{T}^t$, *for* $\mathfrak{T}' = \mathfrak{T}^\times$, *and for* $\mathfrak{T}' = \mathfrak{T}^u$.

(2) *If* $\mathfrak{T}'$ *is a locally convex topology on* E *which is ultrabornological and weaker than* $\mathfrak{T}^u$, *then* $\mathfrak{T}' = \mathfrak{T}^u$.

Proof of (2). Apply DeWilde's closed-graph theorem to $E[\mathfrak{T}'] \to E[\mathfrak{T}^u]$, using the fact that $E[\mathfrak{T}^u]$ is webbed. □

We notice that the above corollary gives a stability property for $(\mathcal{BC})$-spaces which are webbed, as promised in the introduction: If $E[\mathfrak{T}]$ is webbed, then $E[\mathfrak{T}^u]$ is also webbed, hence a $(\mathcal{BC})$-space, i.e., $\mathfrak{T}^u$ is minimal among the ultrabornological topologies on E.

We now consider another property of separated locally convex topologies: $(\alpha) = $ (Baire): "a separated Baire topology". We cannot say too much about the topology $\mathfrak{T}^{\overline{\text{Baire}}}$, $E[\mathfrak{T}]$ a locally convex space, except that, since $(\mathcal{B}) \Rightarrow$ (Baire), we have $(u) \Rightarrow (\overline{\text{Baire}})$, hence $\mathfrak{T}^{\overline{\text{Baire}}} \subset \mathfrak{T}^u$. Using DeWilde's second closed-graph theorem, we obtain the following corollary:

7.3 **Corollary.** *Let* $E[\mathfrak{T}]$ *be a webbed space. Then* $\mathfrak{T}^{\overline{\text{Baire}}} = \mathfrak{T}^u$. $\mathfrak{T}^u$ *is minimal among the separated locally convex topologies on* E *which are the hull topologies defined by a family of Baire spaces, i.e., minimal among the* $\overline{\text{Baire}}$*-topologies* on E.

A webbed $\overline{\text{Baire}}$*-topology is ultrabornological.*

Proof. By DeWilde's second closed-graph theorem, $E[\mathfrak{T}^u]$, being webbed, is a (Baire $\mathcal{C}$)-space. The statements are all trivial consequences of this. □

As a final note, we mention that the above corollary can be viewed as a rather natural companion result to the known result that every strictly webbed Baire space is in fact an (F)-space.

8. **On completeness and closed-graph theorems.** The following three results are reasonably immediate and are intended as introductory. (The fact that the Mackey dual of an (F)-space, or of an ultrabornological space for that matter, is complete follows from the fact that a linear form on such a space is continuous if it is bounded on compact discs.)

8.1 **Proposition.** *Let* $\langle E, E' \rangle$ *and* $\langle F, F' \rangle$ *be dual systems. The following are equivalent*:

(1) *For every closed linear map* $u: F \to E$, u *is weakly continuous.*

(2) *For every dense subspace* H *of* E' *and every weakly continuous map* $v: H \to F'$, v *admits a weakly continuous extension to all of* E'.

8.2 **Theorem.** *Let* E *and* F *be locally convex spaces and suppose the following hypotheses hold*:

(i) *Every dense subspace of the Mackey dual* $E'[\tau(E', E)]$ *of* E *is again a Mackey space.*

(ii) F *is a Mackey space and its Mackey dual* $F'[\tau(F', F)]$ *is complete.*

Then every closed linear map $u: F \to E$ *is continuous.*

8.3 **Corollary.** *Let* E *be the Mackey dual of a metrizable space and let* F *be either an* (F)*-space or the Mackey dual of an* (F)*-space.*

Every closed linear map $u: F \to E$ *is continuous.*

Theorem 8.2 generalizes Corollary 6.2 and Corollary 8.3 extends Corollary

6.3 to the case where the domain may also be the Mackey dual of an (F)-space.

The relationship of Theorem 8.2 to completeness in the dual space is explicit. An examination of Proposition 8.1, with its emphasis on extending a weakly continuous adjoint map clearly suggests the possible utility of completeness conditions on the codomain of the adjoint map, as in the theorem. It is the intent of this section to establish a general method of procedure in closed-graph theory which does utilize such completeness conditions, but in a less direct fashion than in the above theorem. This method of procedure will be based in part on Kōmura's theorem. The results obtained are all analogues in some sense to Adasch's result on infra-(s) spaces.

8.4 **Convention.** For the rest of this section, λ will denote a rule which assigns to each locally convex space $E[\mathfrak{T}]$ two families of sets:

(1) $E\lambda$ is a family of balanced, convex, closed and bounded subsets of E which is increasingly directed, stable under multiplication by scalars, and covers E.

(2) $E'\lambda$ is a family of balanced, convex, weakly closed and bounded subsets of E' which is increasingly directed, stable under multiplication by scalars, and covers E'.

We denote by $\lambda(E', E)$, resp. $\lambda(E, E')$, the structure on E', resp. E, of uniform convergence on sets of $E\lambda$, resp. $E'\lambda$. We subject λ to the following restriction:

(3) If $u: F \to E$ is continuous, then $uF\lambda$ refines $E\lambda$ and $u'E'\lambda$ refines $F'\lambda$.

We notice that (3) implies the following statement:

(3') If $u: F \to E$ is continuous, then u' is $\lambda(E', E)$-$\lambda(F', F)$ continuous and u is $\lambda(F, F')$-$\lambda(E, E')$ continuous. We notice that we have

$$\sigma(E', E) \subset \lambda(E', E) \subset \beta(E', E),$$

$$\sigma(E, E') \subset \lambda(E, E') \subset \beta(E, E').$$

This implies, for instance, that every weakly complete set is λ-complete.

8.5 **Note.** The frame we are in is in many respects symmetric, but not totally so. The families $E\lambda$ and $E'\lambda$ do depend on the given topology on E, as does the continuity condition on linear maps $u: F \to E$. To eliminate this, we could consider a rule, the argument of which is dual pairings $\langle E, E' \rangle$ rather than locally convex spaces E, replacing the continuity condition on linear maps with a weak continuity condition. Such a theory could be studied in our framework by having the rule assign to $E[\mathfrak{T}]$ the families it assigns to $\langle E, E' \rangle$. So the theory we have outlined is the more general one.

Since we will be modifying the topology on E from time to time however, we will have to take care to make it clear which topology is under consideration.

8.6 Lemma. (1) $(\bigoplus_\alpha E_\alpha)\lambda \approx \bigoplus_\alpha (E_\alpha \lambda)$. *Hence,*

$$\lambda\Big(\prod_\alpha E'_\alpha, \bigoplus_\alpha E_\alpha\Big) = \prod_\alpha \lambda(E'_\alpha, E_\alpha).$$

(2) $(\Pi_\alpha E_\alpha)'\lambda \approx \bigoplus_\alpha E'_\alpha \lambda$. *Hence,*

$$\lambda\Big(\prod_\alpha E_\alpha, \bigoplus_\alpha E'_\alpha\Big) = \prod_\alpha \lambda(E_\alpha, E'_\alpha).$$

(3) $(\bigoplus_\alpha E_\alpha)'\lambda$ *refines* $\Pi_\alpha (E'_\alpha \lambda)$ *and is refined by* $\bigoplus_\alpha (E'_\alpha \lambda)$. *Hence,*

$$\bigoplus_\alpha \lambda(E_\alpha, E'_\alpha) \supset \lambda\Big(\bigoplus_\alpha E_\alpha, \prod_\alpha E'_\alpha\Big),$$

$$\lambda\Big(\bigoplus_\alpha E_\alpha, \prod_\alpha E'_\alpha\Big) \supset \prod_\alpha \lambda(E'_\alpha, E'_\alpha)\Big|_{\bigoplus_\alpha E_\alpha}.$$

(4) $(\Pi_\alpha E_\alpha)\lambda$ *refines* $\Pi_\alpha (E_\alpha \lambda)$ *and is refined by* $\bigoplus_\alpha (E_\alpha \lambda)$. *Hence,*

$$\bigoplus_\alpha \lambda\Big(E'_\alpha, E_\alpha\Big) \supset \lambda\Big(\bigoplus_\alpha E'_\alpha, \prod_\alpha E_\alpha\Big),$$

$$\lambda\Big(\bigoplus_\alpha E'_\alpha, \prod_\alpha E_\alpha\Big) \supset \prod_\alpha \lambda(E'_\alpha, E_\alpha)\Big|_{\bigoplus_\alpha E'_\alpha}.$$

(The spaces E_α *are presumed to carry a given locally convex structure and the spaces* $\bigoplus_\alpha E_\alpha$, $\Pi_\alpha E_\alpha$ *presumed to carry the direct sum and product structures, respectively. We identify* $(\bigoplus_\alpha E_\alpha)'$ *with* $\Pi_\alpha E'_\alpha$ *and* $(\Pi_\alpha E_\alpha)'$ *with* $\bigoplus_\alpha E'_\alpha$ *in the canonical way.)*

Proof. We have continuous projections

$$\pi_\alpha : \prod_\beta E_\beta \to E_\alpha, \quad P_\alpha : \bigoplus_\beta E_\beta \to E_\alpha,$$

and continuous injections

$$i_\alpha : E_\alpha \to \prod_\beta E_\beta, \quad j_\alpha : E_\alpha \to \bigoplus_\beta E_\beta.$$

Their adjoints are, respectively, the injections

$$j_\alpha : E'_\alpha \to \bigoplus_\beta E'_\beta, \quad i_\alpha : E'_\alpha \to \prod_\beta E'_\beta$$

and the projections

$$P_\alpha : \bigoplus_\beta E'_\beta \to E'_\alpha, \quad \pi_\alpha : \prod_\beta E'_\beta \to E'_\alpha.$$

Moreover, if B is bounded in $\bigoplus_\beta E_\beta$, resp. $\bigoplus_\beta E'_\beta$, then $P_\alpha(B) = 0$, except for a finite number of indices α. If we combine this with the fact that $u\colon F \to E$ continuous forces $uF\lambda$ to refine $E\lambda$ and $u'E'\lambda$ to refine $F'\lambda$, then we conclude that

$$\Big(\prod_\alpha E_\alpha\Big)\lambda \text{ refines } \prod_\alpha (E_\alpha\lambda),$$
$$\Big(\bigoplus_\alpha E_\alpha\Big)\lambda \text{ refines } \bigoplus_\alpha (E_\alpha\lambda),$$
$$\Big(\bigoplus_\alpha E'_\alpha\Big)\lambda \text{ refines } \bigoplus_\alpha (E'_\alpha\lambda),$$
$$\Big(\prod_\alpha E'_\alpha\Big)\lambda \text{ refines } \prod_\alpha (E'_\alpha\lambda).$$

We now recall that, if L is a convex subset of a vector space and $L_1, \dots, L_n \subset L$, then $L_1 + \dots + L_n = n(L_1/n + \dots + L_n/n) \subset nL$. From this, and from the continuity of the various injections, we conclude that

$$\bigoplus_\alpha (E_\alpha\lambda) \text{ refines } \Big(\prod_\alpha E_\alpha\Big)\lambda,$$
$$\bigoplus_\alpha (E_\alpha\lambda) \text{ refines } \Big(\bigoplus_\alpha E_\alpha\Big)\lambda,$$
$$\bigoplus_\alpha (E'_\alpha\lambda) \text{ refines } \Big(\bigoplus_\alpha E'_\alpha\Big)\lambda,$$
$$\bigoplus_\alpha (E'_\alpha\lambda) \text{ refines } \Big(\prod_\alpha E'_\alpha\Big)\lambda.$$

8.7 **Proposition.** *We consider the following properties of locally convex spaces E:*

(α_1) *$E[\lambda(E, E')]$ is complete.*

(α'_1) *$E'[\lambda(E', E)]$ is complete.*

(α_2) *$E[\lambda(E, E')]$ is quasi-complete.*

(α'_2) *$E'[\lambda(E', E)]$ is quasi-complete.*

(α_3) *Every set in $E\lambda$ is $\lambda(E, E')$-complete.*

(α'_3) *Every set in $E'\lambda$ is $\lambda(E', E)$-complete.*

(α_4) *Every set in $E\lambda$ is $\lambda(E, E')$-compact.*

(α'_4) *Every set in $E'\lambda$ is $\lambda(E', E)$-compact.*

(α_5) *Every set in $E\lambda$ is $\lambda(E, E')$-precompact.*

(α'_5) *Every set in $E'\lambda$ is $\lambda(E', E)$-precompact.*

(α_6) *Every set in $E\lambda$ is $\sigma(E, E')$-compact.*

(α'_6) *Every set in $E'\lambda$ is $\sigma(E', E)$-compact.*

These properties satisfy stability statements as follows:

(1) *Stable under closed subspaces*: (α_1), (α_2), (α_3), (α_6).

(2) *Stable under quotients*: (α'_1), (α'_2), (α'_3), (α'_6).

(3) *Stable under products*: (α_1), (α_2), (α'_2), (α_3), (α'_3), (α_4), (α'_4), (α_5), (α'_5), (α_6), (α'_6).

(4) *Stable under direct sums*: (α'_1), (α_2), (α'_2), (α_3), (α'_3), (α_4), (α'_4), (α_5), (α'_5), (α_6), (α'_6).

(5) *Stable under projective limits*: (α_1), (α_2), (α_3), (α_6).

(6) *Stable under final topologies*: (α'_1), (α'_2), (α'_3), (α'_6).

Proof. (1) Let S be a closed subspace of E. Since $S \to E$ is continuous, it is $\lambda(S, S')$-$\lambda(E, E')$ continuous. Hence, we have

$$\beta(S, S') \supset \lambda(S, S') \supset \lambda(E, E')|_S \supset \sigma(S, S').$$

Moreover, $S\lambda$ refines $E\lambda$. From this we see the stability property holds for all properties involving completeness.

(2) This is simply the dual argument to (1).

(3) and (4) Since the λ-topology on a product or on the dual of a direct sum is just the product of the λ-topologies, since $(\Pi_\alpha E_\alpha)\lambda$ refines $\Pi_\alpha(E_\alpha\lambda)$, since $(\bigoplus_\alpha E_\alpha)'\lambda$ refines $\Pi_\alpha(E'_\alpha\lambda)$, and since all λ-sets are weakly closed, hence λ-closed, it follows that (α_1), (α_2), (α_3), (α_4), (α_5), (α_6) are all stable under products and (α'_1), (α'_2), (α'_3), (α'_4), (α'_5), (α'_6) are all stable under direct sums. In order to complete the argument, we recall that on sums or on the duals of products, the λ-topology has the following properties:

(i) It is stronger than the induced product λ-topology.

(ii) It is weaker than the sum λ-topology.

(iii) Every λ-set is weakly bounded, hence contained in a finite subsum.

(iv) Every λ-bounded set is weakly bounded, hence contained in a finite subsum.

From this, it follows that the induced λ-topology on every λ-set and every λ-bounded set is precisely the induced sum λ-topology. From this, and from (iii), it follows that (α_2), (α_3), (α_4), (α_5), (α_6) are all stable under direct sums and (α'_2), (α'_3), (α'_4), (α'_5), (α'_6) stable under products.

(5) This follows from (1) and (3).

(6) If we first notice that if E has its strongest locally convex topology, then E has *all* the listed properties, we see that we may reduce to the case of locally convex hulls. But, in this case, it suffices to consider direct sums and quotients. So the result follows from (2) and (4).

8.8 Convention. Until further notice, (α) will denote one of the properties (α'_1), (α'_2), (α'_3), (α'_6) introduced in Proposition 7. We concentrate on these because of the fact that they are stable under the formation of final topologies. We note that (α'_2), (α'_3) and (α'_6) have the added virtue of being stable under the formation of arbitrary products. We note also that, in properties (α'_1) and (α'_2), the families $E'\lambda$ are in fact irrelevant (and, along with them, the topologies $\lambda(E, E')$). For this reason, we may, in examples, specify only the family $E\lambda$, in which case one may supply any family $E'\lambda$ which seems reasonable, e.g., let $E'\lambda$ be all finite-dimensional discs.

We will denote by (τ) the property: E is a Mackey space. (τ) is stable under the formation of final topologies and under the formation of products. $(\tau\alpha)$ will denote the property: E is an (α)-space and a Mackey space. $(\tau\alpha)$ is stable under the formation of final topologies and, if (α) is (α'_2), (α'_3) or (α'_6), stable under the formation of products.

If $E[\mathfrak{T}]$ is a locally convex space, we denote, as before, by $\mathfrak{T}^{\alpha}$, resp. $\mathfrak{T}^{\tau\alpha}$, the weakest (α)-topology, resp. $(\tau\alpha)$-topology, on E which is stronger than $\mathfrak{T}$.

8.9 **Proposition.** *Let $E[\mathfrak{T}]$ be a locally convex space and let F be a (δ)-space, where (δ) is either (α) or $(\tau\alpha)$. Let u: $F \to E$ be a continuous linear map.*

(1) *E' is a subspace of $E[\mathfrak{T}^{\delta}]'$ and the inclusion $E' \to E[\mathfrak{T}^{\delta}]'$, as the adjoint of the continuous map $E[\mathfrak{T}^{\delta}] \to E$, is λ-continuous.*

(2) *u': $E' \to F'$ is λ-continuous.*

(3) *u' admits a unique $\sigma(E[\mathfrak{T}^{\delta}]', E)$-continuous extension to $E[\mathfrak{T}^{\delta}]'$, which is in fact the adjoint of the continuous linear map u: $F \to E[\mathfrak{T}^{\delta}]$ and hence is again λ-continuous and weakly continuous.*

(4) *If $(\alpha) = (\alpha'_1)$, then $E[\mathfrak{T}^{\delta}]'$ contains the (Grothendieck) completion of $\widetilde{E'[\lambda(E', E)]}$ and the inclusion $\widetilde{E'_\lambda} \to E[\mathfrak{T}^{\delta}]'$ is continuous with respect to $\widetilde{\lambda(E', E)}$ and $\lambda(E[\mathfrak{T}^{\delta}]', E[\mathfrak{T}^{\delta}])$.*

(5) *If $(\alpha) = (\alpha'_2)$, then $E[\mathfrak{T}^{\delta}]'$ contains the quasi-completion $\overline{E'}$ of $E'[\lambda(E', E)]$ and the inclusion $\overline{E'_\lambda} \to E[\mathfrak{T}^{\delta}]'$ is continuous with respect to $\overline{\lambda(E', E)}$ and $\lambda(E[\mathfrak{T}^{\delta}]', E[\mathfrak{T}^{\delta}])$.*

Proof. (1), (2) and (3) are reasonably clear.

(4) If $\psi \in \widetilde{E'_\lambda}$, then ψ is $\sigma(E, E')$-continuous on each set of $E\lambda$, hence $\sigma(E, E[\mathfrak{T}^{\delta}]')$-continuous on each set of $E[\mathfrak{T}^{\delta}]\lambda$, which refines $E\lambda$. By completeness of $E[\mathfrak{T}^{\delta}]'_\lambda$, it follows that $\psi \in E[\mathfrak{T}^{\delta}]'$. Since $E[\mathfrak{T}^{\delta}]\lambda$ refines $E\lambda$, the continuity statement follows.

(5) Denote by (δ_1), (δ_2) the properties corresponding to (α'_1), (α'_2). Since $E[\mathfrak{T}^{\delta_1}] \to E[\mathfrak{T}^{\delta_2}] \to E[\mathfrak{T}]$ are continuous, it follows that $E'_\lambda \to E[\mathfrak{T}^{\delta_2}]'_\lambda \to E[\mathfrak{T}^{\delta_1}]'_\lambda$ are continuous. Since $E[\mathfrak{T}^{\delta_2}]'_\lambda$ is quasi-complete, it

follows that $E'_\lambda \to E[\mathfrak{T}^{\delta_2}]'_\lambda$ admits a unique continuous extension $\overline{E'_\lambda} \to E[\mathfrak{T}^{\delta_2}]'_\lambda$. Composing this with $E[\mathfrak{T}^{\delta_2}]'_\lambda \to E[\mathfrak{T}^{\delta_1}]'_\lambda$, we obtain a continuous map $\overline{E'_\lambda} \to E[\mathfrak{T}^{\delta_1}]'_\lambda$. We are considering $\overline{E'_\lambda}$ as a subspace of $\widetilde{E'_\lambda}$ and (4) then forces this last map to be $\psi \mapsto \psi$. This forces $\overline{E'_\lambda} \subset E[\mathfrak{T}^{\delta_2}]'$.

8.10 **Note.** A word seems in order here about the meaning of Proposition 9: The completion of a locally convex space can be regarded as the solution of a certain universal mapping problem (as can the quasi-completion). That is, we could define *a* completion of E to be a pair $(i, \widetilde{E})$, where $\widetilde{E}$ is complete, $i: E \to \widetilde{E}$ is continuous and linear, and where, for any other such pair (u, F), *there is a unique* continuous linear map $\tilde{u}: \widetilde{E} \to F$ with $\tilde{u} \circ i = u$. Likewise, the topology $\mathfrak{T}^\delta$ can be viewed as the solution of another universal mapping problem, but this time with the arrows turned around. We can characterize $E[\mathfrak{T}^\delta]$ as follows: $E[\mathfrak{T}^\delta]$ is a (δ)-space and $x \mapsto x$ is a continuous linear map $i: E[\mathfrak{T}^\delta] \to E[\mathfrak{T}]$. If F is any other (δ)-space and if $u: F \to E[\mathfrak{T}]$ is any continuous linear map, then u is also $\mathfrak{T}^\delta$-continuous, i.e., there is a unique continuous linear map $v: F \to E[\mathfrak{T}^\delta]$ such that $i \circ v = u$ ($u = v$, of course).

The point here is that if we take adjoints in this latter exposition, then we have essentially statements (1), (2), and (3) of Proposition 9 and these statements strongly resemble the statements in the first exposition involving completions. The only difference, and it is an important one, is that the mappings involved with E'_λ, F'_λ and $E[\mathfrak{T}^\delta]'_\lambda$ *must be adjoints.* However, taken together with the fact that the property (δ) does, in one way or another, involve some sort of completeness hypothesis, the resemblance between statements (1), (2), (3) of Proposition 9 and the universal mapping properties of completions should justify thinking of $E[\mathfrak{T}^\delta]'_\lambda$ as a kind of completion of E'_λ. Statements (4) and (5) should reinforce that.

As a final reinforcement, we state the following corollary of Proposition 9, offering a special circumstance in which $E[\mathfrak{T}^\delta]'_\lambda$ is, in a very direct sense, a completion of sorts of E'_λ.

8.11 **Corollary.** *Suppose the rule λ depends only on the dual pairing $\langle E, E'\rangle$, i.e., if $\mathfrak{T}$ and $\mathfrak{S}$ are locally convex topologies on E with $E[\mathfrak{T}]' = E[\mathfrak{S}]'$, then λ assigns the same families to $E[\mathfrak{T}]$ and to $E[\mathfrak{S}]$.*

We assume further that, for each locally convex space E, every set in $E\lambda$ is $\sigma(E, E')$-compact.

Let $v: E' \to F'$ be $\lambda(E', E)$-$\lambda(F', F)$ continuous and suppose that F is an (α)-space (that is, that $F'[\lambda(F', F)]$ satisfies one of the completeness conditions (α'_1), (α'_2), (α'_3), (α'_6)).

There is a unique continuous linear map $w: E[\mathfrak{T}^{\alpha}]'_{\lambda} \to F'_{\lambda}$ *which extends* v. ($\mathfrak{T}$ *can be any topology on* E *compatible with the pairing* $\langle E, E' \rangle$.)

Proof. Let $u = v': F \to E$, using the fact that $F = (F')'_{\lambda}$ and $E = (E'_{\lambda})'$. Then u is weakly continuous and Proposition 9 applies. □

We conclude this section by stating two closed-graph theorems, both of which generalize Adasch's theorem, and both of which are direct corollaries of Kōmura's theorem.

8.12 Closed-graph theorem. *Let* $E[\mathfrak{T}]$ *be a locally convex space. The following are equivalent*:

(1) $E[\mathfrak{T}]$ *is an* $(\alpha\mathcal{C})$-*space.*

(2) $\mathfrak{T} \subset \sigma(E, E')^{\alpha}$ *and, for every dense subspace* H *of* E', *we have* $E' \subset E[\sigma(E, H)^{\alpha}]'$.

Proof. (1) ⟹ (2): $E[\sigma(E, H)^{\alpha}] \to E[\mathfrak{T}]$ is closed for all $H \subset E'$ dense.

(2) ⟹ (1): There is a dense subspace H of E' with u $\sigma(E,H)$-continuous, if $u: F \to E$ is closed. But if F is an (α)-space, then u is $\sigma(E, H)^{\alpha}$-continuous. Since $E' \subset E[\sigma(E, H)^{\alpha}]'$, u is $\sigma(E, E')$-continuous, hence $\sigma(E, E')^{\alpha}$-continuous. Since $\mathfrak{T} \subset \sigma(E, E')^{\alpha}$, u is continuous.

8.13 Closed-graph theorem. *Let* $E[\mathfrak{T}]$ *be a locally convex space. The following are equivalent*:

(1) $E[\mathfrak{T}]$ *is a* $(\tau\alpha\mathcal{C})$-*space.*

(2) *For every dense subspace* H *of* E', *we have* $E' \subset E[\tau(E, H)^{\tau\alpha}]'$.

(3) *For every subspace* H *of* E^* *such that* $H \cap E'$ *is dense and* $\tau(E, H)$ *is an* (α)-*topology, we have* $E' \subset H$.

Proof. (2) and (3) are clearly equivalent and (1) ⟹ (2) as in Theorem 13. To prove (2) ⟹ (1), let $u: F \to E$ be closed. There is a dense subspace H of E' with u $\sigma(E, H)$-continuous. Since F is a Mackey space, u is $\tau(E, H)$-continuous. Since F is a $(\tau\alpha)$-space, u is $\tau(E, H)^{\tau\alpha}$-continuous. Since $E' \subset E[\tau(E, H)^{\tau\alpha}]'$, u is $\sigma(E, E')$-continuous. Since F is a Mackey space, u is continuous. □

We remind the reader that, in the above Theorems, a locally convex space F is said to be an (α)-space if it satisfies the appropriate one of the following statements:

F'_{λ} is complete.

F'_{λ} is quasi-complete.

Every set in $F'\lambda$ is $\lambda(F', F)$-complete.

Every set in $F'\lambda$ is $\sigma(F', F)$-compact.

If, in addition, F is a Mackey space, we call F a $(r\alpha)$-space.

The rest of this paper will be devoted to applying the results of this section to particular cases for the rule λ.

9. Algebraic spaces and infra-(s) spaces: A closed-graph theorem for infra-barreled spaces. This section gives the results of that particular case of the theory of §8 got by letting $E\lambda$ be all finite-dimensional discs in E. Then $E'_\lambda = E'_\sigma$.

If $(\alpha) = (\alpha'_1)$, then E is an (α)-space if and only if E'_σ is complete, i.e., $E' = E^*$. In this case, E is a $(r\alpha)$-space if and only if E carries its strongest locally convex topology, in which case E is often called an *algebraic space.* Since, in this case, *every* locally convex space is a $(r\alpha\mathcal{C})$-space, the closed-graph theory is trivial. This is reflected in the fact that statement (2) of Theorem 8.13 translates in this context as: "For every dense subspace H of E', we have $E' \subset E^*$."

The case where $(\alpha) = (\alpha'_2)$ is more interesting, as E is an (α)-space if and only if E'_σ is quasi-complete and E is a $(r\alpha)$-space if and only if E is barreled. In this context, Theorem 8.13 translates directly as Adasch's theorem.

Also interesting is the case that arises when we let $E'\lambda$ be all $\beta(E', E)$-bounded discs in E'. In this case we have $(\alpha'_3) \Leftrightarrow (\alpha'_6)$ and, if we let $(\alpha) = (\alpha'_6)$, then clearly E is an (α)-space if and only if $r(E, E') = \beta^*(E, E')$ and E is a $(r\alpha)$-space if and only if E is infrabarreled.

If $\mathfrak{T}$ is the given topology on E, then $r(E, E[\mathfrak{T}^\alpha]') = \mathfrak{T}^{r\alpha}$ is the weakest infrabarreled topology on E stronger than $\mathfrak{T}$: $E[\mathfrak{T}^\alpha]'_\sigma$ has the property that each of its *strongly* bounded subsets is relatively complete (in fact, relatively compact) and the mapping $E'_\sigma \to E[\mathfrak{T}^\alpha]'_\sigma$ is the solution to the corresponding universal mapping problem (see Corollary 8.11). So $E[\mathfrak{T}^\alpha]'_\sigma$ is a sort of quasi-completion of E'_σ, but we will forebear coining a name for it.

Theorem 8.13 can be restated in this context as follows:

9.1 Corollary. *Let $E[\mathfrak{T}]$ be a locally convex space. The following are equivalent:*

(1) *For every infrabarreled space F and every closed linear map u: $F \to E$, u is continuous.*

(2) *For every subspace H of E^* such that every strongly bounded subset of H is relatively weakly compact and such that $H \cap E'$ is dense, we have $E' \subset H$.*

If these statements are satisfied, then $E[\mathfrak{T}]$ is an $(\mathfrak{N}\mathcal{C})$-space.

10. Closed-graph theorems for the duals of complete spaces. This section gives the results of that particular case of the theory of §8 got by letting

$E\lambda$ be all $\sigma(E, E')$-compact discs in E. Then $E'_\lambda = E'_\tau$. Since, if $\psi \in \widetilde{E'_\tau}$, ψ is $\sigma(E, E')$-continuous on each set of $E\lambda$, it follows that $E\lambda = E[\sigma(E, \widetilde{E'_\tau})]\lambda$. Therefore, $\lambda(\widetilde{E'_\tau}, E)$ is simply the natural topology on $\widetilde{E'_\tau}$, which is, of course, complete. From this, it follows that $\sigma(E, \widetilde{E'_\tau}) = \sigma(E, E')^\alpha$ if $\alpha = (\alpha'_1)$. If $\mathfrak{T}$ is any topology on E compatible with the dual pairing, then $E[\mathfrak{T}^\alpha]' = \widetilde{E'_\tau}$ and the corresponding λ-topology on $\widetilde{E'_\tau}$ is simply its natural topology. Likewise, if $\alpha = (\alpha'_2)$, then $E[\mathfrak{T}^\alpha]' = \overline{E'_\tau}$, the quasi-completion of E'_τ, and the λ-topology on $\overline{E'_\tau}$ is again simply its natural topology.

It seems of interest here to characterize the properties (α) and $(r\alpha)$ viewing E as a dual space when (α) is either (α'_1) or (α'_2): If $(\alpha) = (\alpha'_1)$, resp. (α'_2), then E is an (α)-space if and only if E'_τ is complete, resp. quasi-complete, i.e., if and only if E is the dual of a complete, resp. quasi-complete, space, equipped with some topology compatible with the dual pairing. E is a $(r\alpha)$-space if and only if E is the Mackey dual of a complete, resp. quasi-complete, space.

It should be noted also that here, as with any rule which depends only on the dual pairing $\langle E, E'\rangle$, if $\mathfrak{T}$ and $\mathfrak{S}$ are compatible topologies on E, then, even though $\mathfrak{T}^\alpha$ may not be the same as $\mathfrak{S}^\alpha$, we do have $\mathfrak{T}^{r\alpha} = \mathfrak{S}^{r\alpha}$ and, moreover we may have $E[\mathfrak{T}^\alpha]'_\lambda = E[\mathfrak{S}^\alpha]'_\lambda = E[\mathfrak{T}^{r\alpha}]'_\lambda$. That is, the spaces are the same, as are the topologies on them.

The following two results are rather immediate restatements of Theorem 8.13 in this context:

10.1 Corollary. *Let $E[\mathfrak{T}]$ be a locally convex space. The following are equivalent:*

(1) *If F is the Mackey dual of a complete space and if $u: F \to E$ is a closed linear map, then u is continuous.*

(2) *For every dense subspace H of E', if $H_\tau = H[\tau(H, E)]$, then $E' \subset \widetilde{H_\tau}$.*

10.2 Corollary. *Let $E[\mathfrak{T}]$ be a locally convex space. The following are equivalent:*

(1) *If F is the Mackey dual of a quasi-complete space and if $u: F \to E$ is a closed linear map, then u is continuous.*

(2) *If H is a dense subspace of E', and if $H_\tau = H[\tau(H, E)]$, then $E' \subset \overline{H_\tau}$.*

Note. Compare the first result with Theorem 8.2. Compare it also with Corollary 6.1. That statement says that, if F is ultrabornological then a closed map $u: F \to E$ is continuous provided $f(K)$ is bounded for all $f \in E'$ and all $K \subset E$, K a $\sigma(E, H)$-compact disc. Here, we weaken the condition on

F to F'_τ complete and we must strengthen the condition on E and now assume that f is $\sigma(E, H)$-continuous on K for each $\sigma(E, H)$-compact disc K.

It was mentioned in the introduction that Adasch's theorem yields a rather easy proof of Pták's closed-graph theorem, since it is not difficult to show that an infra-Pták space is an infra-(s) space. This lends some interest to the following result:

10.3 **Corollary.** *Let E be a semireflexive infra-Pták space and let F be the Mackey dual of a quasi-complete space.*

If u: $F \to E$ is a closed linear map, then u is continuous.

Proof. Let H be a dense subspace of E'. It suffices to show that $E' \subset \overline{H_\tau}$. We use the following lemma:

10.4 **Lemma.** *Let E be a locally convex space and denote by $\mathfrak{M}$ the family of all balanced, convex, $\sigma(E', E)$-closed, equicontinuous subsets of E'. The following are equivalent:*

(1) *E is a semireflexive infra-Pták space.*

(2) *For every dense subspace K of E' such that $K \cap M$ is $\beta(E', E)$-closed in E' for all $M \in \mathfrak{M}$, we have $K = E'$.*

Proof. (1) $\Rightarrow$ (2) is clear since $E'_\tau = E'_\beta$.

(2) $\Rightarrow$ (1): It suffices to show $E = (E'_\beta)'$, since then $E'_\tau = E'_\beta$ and the condition then reduces to the condition that E be an infra-Pták space.

Let $\psi \in (E'_\beta)'$ and suppose $\operatorname{Ker} \psi$ is dense in E'_σ. We must show $\psi = 0$. Let $K = \operatorname{Ker} \psi$. Then K is strongly closed. So $K \cap M$ is strongly closed for all $M \in \mathfrak{M}$. So $K = E'$. So $\psi = 0$.

Proof of corollary, continued. We consider the uniform structure $\mathfrak{U}$ on E^* of uniform convergence on $\sigma(E, H)$-compact discs. Since E is an infra-Pták space, hence a $(\mathcal{B}\mathcal{C})$-space, all such discs are bounded in E, and so $\mathfrak{U}$ induces a locally convex structure on E' which is stronger than $\sigma(E', E)$ and weaker than $\beta(E', E)$. Since each set in $\mathfrak{M}$ is a $\sigma(E', E)$-compact disc, it is therefore $\mathfrak{U}$-complete and $\mathfrak{U}$-bounded in E'.

Let $K = \overline{H_\tau} \cap E'$. If $M \in \mathfrak{M}$, then $M \cap K = M \cap \overline{H_\tau}$. Since M is $\mathfrak{U}$-complete and bounded and since $\overline{H_\tau}$ is $\mathfrak{U}$-quasi-complete, it follows that $M \cap \overline{H_\tau}$ is $\mathfrak{U}$-complete, hence $\mathfrak{U}$-closed in E', hence $\beta(E', E)$-closed in E'. Hence $K = E'$, i.e., $E \subset \overline{H_\tau}$. □

Note. In Corollary 10.3, there is no hope of removing entirely the hypothesis that E be semireflexive, since, by Mahowald's theorem, if for every $(\mathcal{B})$-space E and every closed linear map u: $F \to E$, u is continuous, then it follows that F is barreled. (See Horváth [5, pp. 303–304].) But it is quite possible for F'_τ to be quasi-complete without F_τ being barreled, i.e., without F'_σ being quasi-complete. (That is, not every quasi-complete space is semi-reflexive.)

11. Closed-graph theorems and completeness in the polar topology. Again, this section gives the results of a special case for the rule λ of §8.

This example, unlike the previous two involves a rule which does *not* depend simply on dual pairings. We let $E\lambda$ be all precompact discs of E. Then E'_λ is the space which, following the notation of Dazord and Jourlin [2], we denote by E'_p. (If $\mathcal{T}$ is the given topology on E, then the topology $\lambda(E', E)$ is often denoted by $\mathcal{T}^0$ and called the *polar topology* to $\mathcal{T}$. See Köthe [8, §21,7]. The fundamental fact, due essentially to Ascoli's theorem, is that a bounded subset of E' is precompact with respect to $\mathcal{T}^0$ if and only if it is equicontinuous on every precompact disc in E. Hence, all equicontinuous subsets of E' are $\mathcal{T}^0$-precompact and the $\sigma(E', E)$-closure of a $\mathcal{T}^0$-precompact set is again $\mathcal{T}^0$-precompact.) We also specify a family $E'\lambda$, namely all $\lambda(E', E)$-precompact and $\sigma(E', E)$-closed discs in E'. Since every $\lambda(E', E)$-precompact subset of E' is contained in such a set, the space $E[\lambda(E, E')]$ is simply E, with the topology induced by the canonical injection $E \to (E'_p)'_p$. We denote this space by E_p. (The topology on E_p is often denoted $\mathcal{T}^{00}$. We have $\mathcal{T} \subset \mathcal{T}^{00}$.)

We first mention the properties (α'_3) and (α'_6) : They are equivalent and reduce to the statement that E'_p is *polar-semireflexive*, i.e., that every precompact subset of E'_p is relatively compact.

11.1 **Lemma.** *Let* $(\alpha) = (\alpha'_1)$: "E'_p *is complete.*"

Let $\widetilde{E'_p} \subset E^*$ *denote the Grothendieck completion of* E'_p. *We denote by* $E_{\tilde{p}}$ *the space* E, *with the structure of uniform convergence on the compact discs of* $\widetilde{E'_p}$. *Then* $E_{\tilde{p}}$ *is a locally convex space and we have continuous maps* $E_{\tilde{p}} \to E_p \to E$ *with all these spaces sharing the same precompact sets. We therefore have topological embeddings* $E'_p \to (E_p)'_p \to (E_{\tilde{p}})'_p = \widetilde{E'_p}$. *Finally, we have* $\widetilde{E'_p} = E[\mathcal{T}^\alpha]'$.

Proof. Consider the space $\widetilde{E'_\lambda} = \widetilde{E'_p}$. Since it consists precisely of all linear forms in E^* which are continuous on each precompact subset of E, we may appeal again to Ascoli's theorem and conclude that, considering $\widetilde{E'_p}$ as a locally convex space in its own right, the topology on E of uniform convergence on sets in $(\widetilde{E'_p})\lambda$, i.e., on all precompact (hence compact) discs in $\widetilde{E'_p}$, is the strongest locally convex topology on E which coincides with the given topology $\mathcal{T}$ on every set of $E\lambda$. The dual space of E with respect to this topology is clearly $\widetilde{E'_p}$. If we denote E, together with this topology by $E_{\tilde{p}}$, then we have continuous maps $E_{\tilde{p}} \to E_p \to E$. Moreover, since the topology of $E_{\tilde{p}}$ agrees with that of E on all the precompact discs of E, and since the natural (completion) topology on $\widetilde{E'_p}$ is uniform convergence on such sets, we have $\widetilde{E'_p} = (E_{\tilde{p}})'_p$. So we have topological embeddings $E'_p \to (E_p)'_p \to (E_{\tilde{p}})'_p = \widetilde{E'_p}$. The point to note however is that $(E_{\tilde{p}})'_p$ is complete. From this, it follows that if $(\alpha) = (\alpha'_1)$ then we have a continuous injection $E_{\tilde{p}} \to E[\mathcal{T}^\alpha]$. Hence $E[\mathcal{T}^\alpha]' \subset (E_{\tilde{p}})' = \widetilde{E'_p}$. Since $\widetilde{E'_p} \subset E[\mathcal{T}^\alpha]'$ by Proposition 8.9, we have $\widetilde{E'_p} = E[\mathcal{T}^\alpha]'$. Moreover, since E, $E[\mathcal{T}^\alpha]$ and $E_{\tilde{p}}$ all have the same precompact sets, we have $\widetilde{E'_p} = (E_{\tilde{p}})'_p = E[\mathcal{T}^\alpha]'_p$. □

A note seems in order here. If E is infinite dimensional and has the topology $\mathcal{T} = \sigma(E, E^*)$, then, even though $E[\mathcal{T}^\alpha]' = (E_{\tilde{p}})'$, we do not have $\mathcal{T}^\alpha$ coinciding with the topology on $E_{\tilde{p}}$. Indeed, $\mathcal{T}^\alpha = \mathcal{T}$ and the latter topology is the strongest locally convex topology on E.

At this stage, in this context, only a restatement of Theorem 8.12 is useful. We give it in what is essentially the case where E carries a weak topology. In this case, $E'_p = E'_\beta$.

11.2 **Corollary.** *Let E be a locally convex space. The following are equivalent:*

(1) *For every locally convex space F such that F'_p is complete and every closed linear map u: $F \to E$, u is weakly continuous.*

(2) *For every dense subspace H of E', we have $E' \subset \widetilde{H_\beta}$.*

Note. The above result should be compared with the results on $(\mathfrak{NC})$-spaces: E is an $(\mathfrak{NC})$-space if, for every dense subspace H of E', every $\sigma(E, H)$-bounded set $B \subset E$, and every $f \in E'$, $f(B)$ is bounded. If we wish to weaken the condition on the domain space F to F'_β complete and get every closed linear map u: $F \to E$ weakly continuous, then we must strengthen the condition on $f \in E'$ to f $\sigma(E, H)$-continuous on each $\sigma(E, H)$-bounded set $B \subset E'$.

The next lemma follows easily from Lemma 11.1:

11.3 **Lemma.** *Let $(\alpha) = (\alpha'_2)$: "E'_p is quasi-complete."*

We denote E, together with the topology of uniform convergence on the compact discs of $\overline{E'_p}$, by $E_{\bar{p}}$. We have continuous linear maps $E_{\tilde{p}} \to E_{\bar{p}} \to E_p \to E$ and all the spaces involved share the same precompact sets. We conclude then that $\overline{E'_p} = (E_{\bar{p}})'_p$ and hence that $E[\mathcal{T}^\alpha]'_p = (E_{\bar{p}})'_p = \overline{E'_p}$.

Again, we get a restatement of Theorem 8.12 in this context:

11.4 **Corollary.** *Let E be a locally convex space. The following are equivalent:*

(1) *For every locally convex space F such that F'_p is quasi-complete and every closed linear map $u: F \longrightarrow E$, u is weakly continuous.*

(2) *For every dense subspace H of E', we have $E' \subset \overline{H_\beta}$.*

We denote by $\check{E}'_p$ the smallest subspace of $\overline{E'_p}$ containing E' which is polar-semireflexive, i.e., in which every precompact set is relatively compact. The next lemma follows easily from Lemma 11.1.

11.5 **Lemma.** *Let $(\alpha) = (\alpha'_3) \Longleftrightarrow (\alpha'_6)$: "$E'_p$ is polar-semireflexive."*

We denote by $E_{\check{p}}$ the locally convex space obtained by equipping E with the structure on uniform convergence on the compact sets of $\check{E}'_p$. We have $(E_{\check{p}})' = \check{E}'_p$ and we have continuous maps $E_{\tilde{p}} \longrightarrow E_{\overline{p}} \longrightarrow E_{\check{p}} \longrightarrow E_p \longrightarrow E$, all the spaces involved sharing the same precompact sets. From this, we conclude that $\check{E}'_p = (E_{\check{p}})'_p$ and so $(E_{\check{p}})'_p$ is polar-semireflexive. Hence we have a continuous map $E_{\check{p}} \longrightarrow E[\mathfrak{T}^\alpha]$, both spaces sharing the same precompact sets, and so we have topological embeddings $E'_p \longrightarrow E[\mathfrak{T}^\alpha]'_p \longrightarrow \check{E}'_p \longrightarrow \overline{E'_p} \longrightarrow \widetilde{E'_p}$. But $E[\mathfrak{T}^\alpha]'_p$ is polar-semireflexive. Hence we have $E[\mathfrak{T}^\alpha]'_p = \check{E}'_p$.

Again, we get a restatement of Theorem 8.12 in this context.

11.6 **Corollary.** *Let E be a locally convex space. The following are equivalent:*

(1) *For every locally convex space F such that F'_p is polar-semireflexive and every closed linear map $u: F \longrightarrow E$, u is weakly continuous.*

(2) *For every dense subspace H of E', we have $E' \subset \check{H}_\beta$.*

Each of the three closed-graph theorems stated so far might just as well have been stated in terms of a domain space F for which F'_β satisfies a given completeness condition. Generally speaking this is not a very strong condition to impose on F and it is to be expected that, in order to ease the task for finding a suitable space E as the codomain, the condition on F should be strengthened. So we shift our attention from (α) to $(r\alpha)$.

Passing from (α) to $(r\alpha)$ is a nonnegligible process in this case. The reason lies in the difficulty of giving a satisfying description of $E[\mathfrak{T}^{r\alpha}]'$. The following two lemmas probably will not help much, but they seem to be all that is available. The first could just as well have been stated in §8. After the second, we will proceed immediately with closed-graph theorems.

11.7 **Lemma.** *Let $E[\mathfrak{T}]$ be a locally convex space. Let $\mathfrak{T}_0 = \mathfrak{T}$. If n is an even ordinal, let $\mathfrak{T}_{n+1}$ be the Mackey topology of $\mathfrak{T}_n$. If n is an odd* *ordinal, let* $\mathfrak{T}_{n+1} = \mathfrak{T}^{\alpha}_n$. *If* $n > 0$ *is a limit ordinal, let* $\mathfrak{T}_n = \bigcup_{m<n} \mathfrak{T}_m$. *One obtains an increasing transfinite sequence of locally convex topologies which must eventually be stationary and one has* $\mathfrak{T}^{\tau\alpha} = \mathfrak{T}_n$ *for the first* n *such that* $\mathfrak{T}_n = \mathfrak{T}_{n+1}$. *(One also has* $\mathfrak{T}^{\tau\alpha} = \mathfrak{T}_m$ *for all* $m \geq n$.)

11.8 **Lemma.** *Let* E *be a locally convex space. To save time, we introduce the symbol* $\square$, *which will stand for* $\sim$, *resp.* $\neg$, *resp.* $\smile$, *as* (α) *is* (α'_1), *resp.* (α'_2), *resp.* (α'_3). *We let* $E_0 = E$. *If* n *is an even ordinal, we let* E_{n+1} *be* $(E_n)_\tau$. *If* n *is an odd ordinal, we let* E_{n+1} *be* $(E_n)_{\overset{\square}{p}}$. *If* $n > 0$ *is a limit ordinal, we let* $E_n = \varprojlim_{m<n} E_m$. *We notice in this case that a set in* E_n *is precompact if and only if it is precompact in* E_m *for all* $m < n$. *From this, it follows that the construction may be dualized as follows: Let* $E'_1 = (E_\tau)'_p$. *If* n *is an* $\underset{\square}{\text{even}}$ *ordinal, we let* E'_{n+1} *be* $((E_n)_\tau)'_p$. *If* n *is an odd ordinal, we let* E'_{n+1} *be* $\overset{\square}{E}{}'_n$. *If* $n > 0$ *is a limit ordinal, we let* $E'_n = \varinjlim_{m<n} E'_m$ *(as a locally convex space! All the spaces* E'_n *have topologies, either polar topologies or topologies induced from completions.) We notice that we have continuous maps* $E_n \leftarrow E_m$ *and* $E'_n \rightarrow E'_m$ *if* $n \leq m$. *Notice also that, if* $n > 0$ *is a limit ordinal, then* $E'_n = \varinjlim_{m<n} E'_m$ *is in fact just* $(E_n)'_p$, *by the above remark about precompact sets.*

Then, $E[\mathfrak{T}^{\tau\alpha}] = \varprojlim_n E_n$ *and* $E[\mathfrak{T}^{\tau\alpha}]'_p = \varinjlim_n E'_n$.

Proof. We return to the notation of the previous lemma. We denote $E[\mathfrak{T}_n]$ by E_n. We then have a continuous map $(E_n)_{\tilde{p}} \rightarrow E[\mathfrak{T}^{\alpha}_n]$, resp. $(E_n)_{\bar{p}} \rightarrow E[\mathfrak{T}^{\alpha}_n]$, resp. $(E_n)_{\breve{p}} \rightarrow E[\mathfrak{T}^{\alpha}_n]$, according as (α) is (α'_1), resp. (α'_2), resp. (α'_3). Moreover, both of the spaces involved share the same dual and hence share the same Mackey topology. Therefore, we could just as well have let $\mathfrak{T}_{n+1}$ be the topology on $(E_n)_{\tilde{p}}$, resp. $(E_n)_{\bar{p}}$, resp. $(E_n)_{\breve{p}}$, as $\mathfrak{T}^{\alpha}_n$. With these remarks, the present lemma is transparent.

11.9 **Corollary.** *Let* (δ) *be one of the following properties of locally convex spaces* F:

(1) F *is a Mackey space and* F'_p *is complete.*

(2) F *is a Mackey space and* F'_p *is quasi-complete.*

(3) F *is a Mackey space and* F'_p *is polar-semireflexive.*

Let E *be a locally convex space. The following are equivalent:*

(a) E *is a* $(\delta\mathcal{C})$-*space.*

(b) *For every dense subspace* H *of* E', *we have* $E' \subset E[\tau(E, H)^{\delta}]'$.

A sufficient condition for E *to satisfy these statements is the following, where, for each dense subspace* H *of* E', H_p *denotes the space* $E[\tau(E, H)]'_p$:

(1) *For each dense subspace* H *of* E', $E' \subset \widetilde{H_p}$.

(2) *For each dense subspace* H *of* E', $E' \subset \overline{H}_p$.

(3) *For each dense subspace* H *of* E', $E' \subset \breve{H}_p$.

If E *satisfies the statements corresponding to* (1), *then* E *is an* $(\mathcal{NC})$-*space*..

Proof. (a) $\Leftrightarrow$ (b) is just a restatement of Theorem 8.13 in this context. The sufficiency of the indicated conditions follows immediately from the lemma. The remark about $(\mathcal{NC})$-spaces is treated in the following note.

Note. We conclude this section by saying something about the relationship between various polar topologies: If $\mathcal{T}$ is the given topology on E, then since $\sigma(E, E') \subset \mathcal{T} \subset \tau(E, E')$, we have $\tau(E, E')^0 \subset \mathcal{T}^0 \subset \sigma(E, E')^0$. Since we also have $\sigma(E', E) \subset \tau(E, E')^0$ and $\sigma(E, E')^0 = \beta(E', E)$, it follows that, for any subset of E', we have $\tau(E, E')^0$-completeness $\Rightarrow$ $\mathcal{T}^0$-completeness $\Rightarrow$ $\sigma(E, E')^0 = \beta(E', E)$-completeness. Moreover, if $B \subset E$ is bounded and if $S \subset E'$ with $\sup_{x \in B, u \in S} |u(x)| = +\infty$, then we may choose a sequence $\{x_n\}$ in B with $\sup_{u \in S} |u(x_n)| \geq n^2$ and conclude that $\sup_{u \in S} |u(x_n/n)| \geq n$. From this, it follows that S is unbounded on the τ-convergent sequence $\{x_n/x\}$ and we conclude that $\tau(E, E')^0$, $\mathcal{T}^0$ and $\sigma(E, E')^0 = \beta(E', E)$ all share the same bounded sets. From this, it follows that if $\tau(E, E') = \beta^*(E, E')$, i.e., if every $\beta(E', E)$-bounded set is relatively $\sigma(E', E)$-compact, then $(E_\tau)'_p$, E'_p and $(E_\sigma)'_p$ are all quasi-complete. Of course, we also have the relations $(E_\tau)'_p$ polar-semireflexive $\Rightarrow$ E'_p polar-semireflexive $\Rightarrow$ $(E_\sigma)'_p$ polar-semireflexive; $(E_\tau)'_p$ quasi-complete $\Rightarrow$ E'_p quasi-complete $\Rightarrow$ $(E_\sigma)'_p$ quasi-complete; $(E_\tau)'_p$ complete $\Rightarrow$ E'_p complete $\Rightarrow$ $(E_\sigma)'_p$ complete. These relations can be deduced directly from the above comparison of topologies, or read off the relation $\sigma(E, E')^\alpha \subset \mathcal{T}^\alpha \subset \tau(E, E')^\alpha$ for the various choices of (α). We remark that the observation pertaining to the case $\tau(E, E') = \beta^*(E, E')$, i.e., when E_τ is infrabarreled, in which case E'_p is quasi-complete, can be strengthened when E is bornological: In this case, a form in E^* is actually in E' if and only if it is bounded on null-sequences. From this, it follows that E'_p is complete.

12. **Closed graph theorems and completeness with respect to compact convergence.** This section gives the last particular case which we shall give of the theory of §8.

This example is much the same as the last, except that, instead of initiating the rule λ with a "precompactology", we initiate it with a "compactology": $E\lambda$ is taken to be all compact discs in E and E'_c will denote $E'[\lambda(E', E)]$. $E'\lambda$ is taken to be all precompact discs in E'_c and we will denote $E[\lambda(E, E')]$ by E_{cp}. $\widetilde{E'_c}$, $\overline{E'_c}$, and $\breve{E'_c}$ are defined as in §11. We have continuous maps

$$\begin{array}{ccccccc} E'_p & \to & \check{E'_p} & \to & \overline{E'_p} & \to & \widetilde{E'_p} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E'_c & \to & \check{E'_c} & \to & \overline{E'_c} & \to & \widetilde{E'_c} \end{array}$$

Just as in §11, we can obtain corresponding continuous maps

$$\begin{array}{ccccccccc} E_{\widetilde{cp}} & \to & E_{\overline{cp}} & \to & E_{\check{cp}} & \to & E_{cp} & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ E_{\tilde{p}} & \to & E_{\overline{p}} & \to & E_{\check{p}} & \to & E_p & \to & E \end{array}$$

with the spaces on the top row sharing the same compactology (having the same compact discs). From this, just as in §11, it follows that, if $(\alpha) = (\alpha'_1)$, then $E[\mathfrak{T}^\alpha]'_c = (E_{\widetilde{cp}})'_c = \widetilde{E'_c}$; if $(\alpha) = (\alpha'_2)$, then $E[\mathfrak{T}^\alpha]'_c = (E_{\overline{cp}})'_c = \overline{E'_c}$; if $(\alpha) = (\alpha'_3) \Leftrightarrow (\alpha'_6)$, then $E[\mathfrak{T}^\alpha]'_c = (E_{\check{cp}})'_c = E'_c$. The example given in case of polar topologies shows here also that though we have continuous maps $E_{\widetilde{cp}} \to E[\mathfrak{T}^\alpha]$, resp. $E_{\overline{cp}} \to E[\mathfrak{T}^\alpha]$, resp. $E_{\check{cp}} \to E[\mathfrak{T}^\alpha]$, according as (α) is (α'_1), resp. (α'_2), resp. (α'_3), and though the spaces share the same dual and the same compactology, they need not be the same. Also, the process of "constructing" the topology $\mathfrak{T}^{r\alpha}$ and its dual space $E[\mathfrak{T}^{r\alpha}]'_c$ proceeds in the same fashion as that in §11 and there seems to be no reason for the author to suppose that the construction should have more or fewer drawbacks than the other one.

The results of varying the topology on E in this case are similar to those in §11, but there are differences: The principal difference is that $(E_\sigma)'_p = E'_\beta$, whereas $(E_\sigma)'_c = E'_\tau$. Since, in any case, $\sigma(E', E) \subset \lambda(E', E) \subset \tau(E, E')$, it follows that the spaces $(E_\sigma)'_c$, E'_c and $(E_\tau)'_c$ share the same bounded sets, but, unlike the case for E'_p, these sets need not be strongly bounded. However, we can state the analogue to the statement we gave then: If E_τ is barreled, then E'_c is quasi-complete, since every bounded set in E'_c is $\sigma(E', E)$-bounded, hence relatively $\sigma(E', E)$-compact. In the case where E is ultrabornological, this statement can be strengthened: A linear form in E^* is in E' if and only if it is bounded on the compact discs of E. Hence it follows that E'_c is complete in this case.

It is quite probably the case that the example in this section is much more important than that of §11, since the condition that E'_c satisfy some sort of completeness statement is much more difficult to meet than the condition that E'_p satisfy the corresponding statement, but the condition is still inclusive enough to be satisfied by any ultrabornological space. The theorems of the previous section generalize our theorems about $(\mathcal{NC})$-spaces, a study that is already over-general. Nonetheless, the last section is a natural prelude to this one, and, given that section, any attempt to develop the material of this section with such explicitness would only be repetitious.

It is also interesting that this section directly generalizes §10, since the theory here reduces to that of that section if the topology on all spaces involved is presumed weak. We make that explicit in the following restatement of Theorem 8.12 in this context:

12.1 **Corollary.** *Let $\langle E, E'\rangle$ be a dual system.*

(1) *The following are equivalent*:

(a) *For every locally convex space F such that F'_c is complete and every closed linear map u: $F \to E$, u is weakly continuous.*

(b) *For every dense subspace H of E', we have $E' \subset \widetilde{H_\tau}$.*

(2) *The following are equivalent*:

(a) *For every locally convex space F such that F'_c is quasi-complete and every closed linear map u: $F \to E$, u is weakly continuous.*

(b) *For every dense subspace H of E', we have $E' \subset \overline{H_\tau}$.*

(The reader should by now have looked back at Corollaries 10.1 and 10.2. A moment's reflection will assure him that, to this point, the content of this corollary is *precisely* that of those corollaries, but is looked at from a slightly different perspective.)

(3) *The following are equivalent*:

(a) *For every locally convex space F such that F'_c is polar-semireflexive and every closed linear map u: $F \to E$, u is weakly continuous.*

(b) *For every dense subspace H of E', we have $E' \subset \breve{H}_\tau$.*

If the statements of (1) *are satisfied, then E_τ is a $(\mathcal{BC})$-space.*

The following is the restatement of Theorem 8.13 in this context.

12.2 **Corollary.** *Let (δ) be one of the following properties of locally convex spaces F*:

(1) *F is a Mackey space and F'_c is complete.*

(2) *F is a Mackey space and F'_c is quasi-complete.*

(3) *F is a Mackey space and F'_c is polar-semireflexive.*

Let E be a locally convex space. The following are equivalent:

(a) *E is a $(\delta\mathcal{C})$-space.*

(b) *For every dense subspace H of E', we have $E' \subset E[\tau(E, H)^\delta]'$.*

A sufficient condition for E to satisfy these statements is the following, where, for each dense subspace H of E', H_c denotes the space $E[\tau(E, H)]'_c$.

(1) *For each dense subspace H of E',* $E' \subset \widetilde{H_c}$.

(2) *For each dense subspace H of E',* $E' \subset \overline{H_c}$.

(3) *For each dense subspace H of E',* $E' \subset \breve{H}_c$.

If E satisfies the statements corresponding to (1), *then E is a* $(\mathcal{BC})$-*space.*

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